# Lecture notes on Electrodynamics–118120

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# Contents

# Chapter 1

# Tensor calculus

Tensor calculus allows us to write equation without making a commitment to a coordinate system.

## 1.1 The geometry of space time

Minkowski space-time is the stage on which classical electrodynamics takes place. Before describing the geometry of space-time lets us collect the tools we need from geometry.

In constructing physical theories we need to know some facts about the system we consider. For example, how many particles we have of different species, in a given region of space-time. These things are scalars. They only depend on our ability to count. We also need various physical constants such as $e$, $\hbar$, and the masses of varous particles etc. All of these are God given scalars. We also need to know something about the space we live in and how to measure distances between events in space-time. For the purpose of this course, space-time is Minkowski and space is Euclidean. We take this to be another God given fact.

In Minkowski space, we can distribute copies of the same meter stick and the same clocks everywhere. (This depends on the fact that Minkowsky space is homegeneous and isotropic). It is then an experimental fact that with such sticks and clocks, the velocity of light $c$ is a universal constant. In this sense $c$ can be viewed as a scalar.

We are still free to choose any (curvilinear in general) coordinate systems to describe the points of Minkowski space. The coordinate system need not be inertial. The price we pay is that meter sticks and clocks that are stationary in such a general coordinate system may not remain synchronized.

**Remark 1.1** (Euclid, Gauss and Einstein)**.** *Euclid took it for granted that physical space is Euclidean. In a Euclidean world the angles of triangles sum up to* $\pi$*. The first to seriously entertain the possibility that space need not be Euclidean was Gauss. When one says that the world is to a good approximation Euclidean one means that the deviations from* $\pi$ *are small. Gauss who essentially* *invented lad surveying tested this. Later Einstein taught us that space-time is actually curved and there are many physical tests of this. This is a another story.*

### 1.1.1 The metric tensor

In the Euclidean plane consider a Cartesian coordinate system $(x, y)$, and a polar coordinate systems $(r, \theta)$.

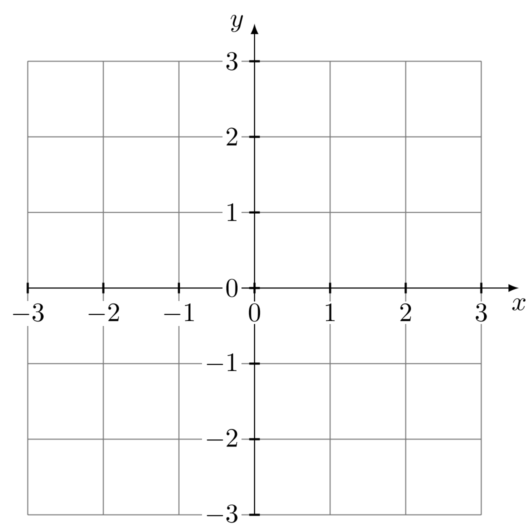


Figure 1.1: Cartesian coordinates for the Euclidean plane

The distance between two neighboring points using a standard meter is denoted $d\ell$: It does not depend on the choice of coordinates

$$(d\ell)^2 = (dx)^2 + (dy)^2 = (dr)^2 + r^2\,(d\theta)^2 = \sum_{ij} g_{ij} dx^i dx^j$$

$g$ is called the metric tensor, also known as the Riemann metric tensor. It is a *second rank tensor*: It has two indices and is symmetric $g_{ij} = g_{ji}$. When $g$ is a diagonal, the coordinates are called orthogonal.

Note that the coordinates have upstairs indices while the metric has them downstairs.

A one to one mapping

$$\{x^1, x^2\} \mapsto \{\xi^1, \xi^2\}$$

is a coordinate transformation. In Euclidean one can pick cooridnates $\xi^j$ that are curvilinear. For example, polar coordinates in the plane

$$x = \rho\cos\theta, \quad y = \rho\sin\theta,$$

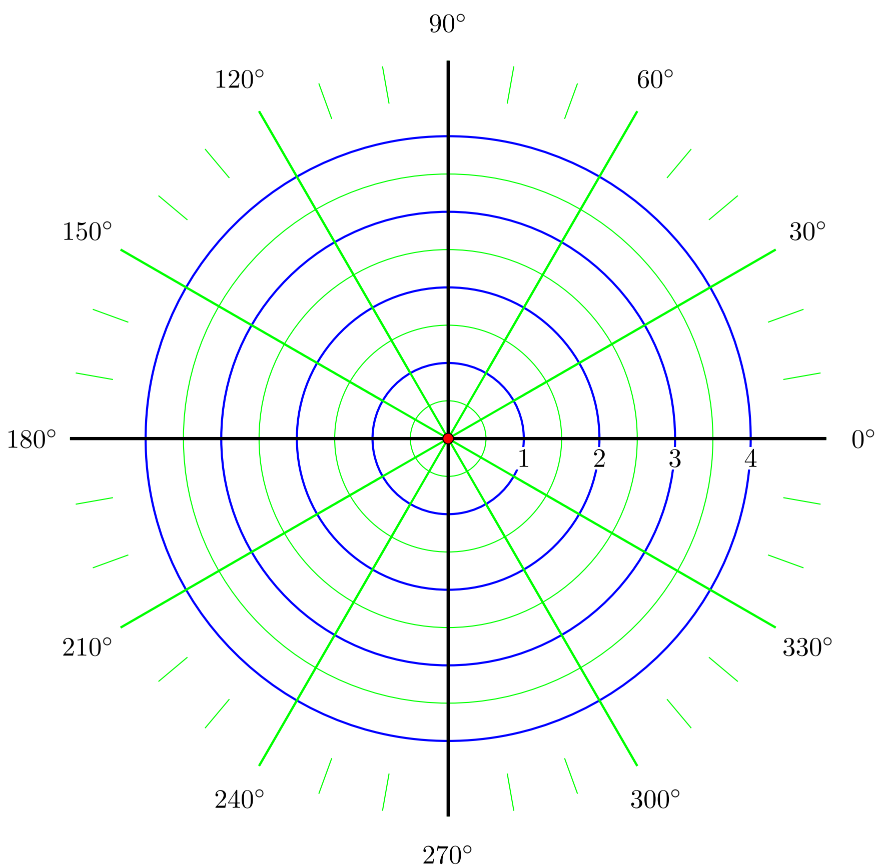


Figure 1.2: Polar coordinates for the Euclidean plane

where the domain is

$$-\infty < x, y < \infty, \quad 0 \leq \rho < \infty, \quad -\pi < \theta \leq \pi$$

The mapping is $1-1$ except for the origin where the point $(x=0, y=0)$ maps to a line $(\rho = 0, \theta)$. You see this in the pictures where the $\theta$ coordinates collapse at the origin. This is a coordinate singularity, which does not reflect any bad physical features, the space is still nice and smooth there.

The metric in the $\xi$ coordinates, which we denote by $\gamma$ is, the chain rule, (and Pythagoras),

$$(d\ell)^2 = \sum (dx^i)^2 = \sum \gamma_{jk} d\xi^j d\xi^k, \quad \gamma = \Lambda^t \Lambda, \quad {\Lambda^i}_j = \frac{\partial x^i}{\partial \xi^j} \tag{1.1}$$

where $i$ is the row index and $j$ the column index of the matrix $\Lambda$. For polar coordinates we get

$$\Lambda = \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

Note that

$$\det \Lambda = \rho \geq 0$$

so the map is invertible (and orientation preserving) except for $\rho = 0$. The vanishing of the metric at $\rho = 0$ is a reflection of a coordinate singularity.

### 1.1.2 Einstein summation convention

This is a short hand which says: Sum over pairs of up-down indices. For example

$$\sum g_{ij}dx^i dx^j = g_{ij}dx^i dx^j$$

It is also called *contraction* of indices.

**Remark 1.2** (Dummy indices)**.** *Summation indices are sometimes called running and sometimes dummy. They can be relabeled freely*

$$g_{ja}v^a = g_{jb}v^b$$

**Remark 1.3** (Warning)**.** *If you get an equation where the indices are not nicely paired, such as*

$$v_a u_a, \quad v_a u^a w_a$$

*it is a good idea to search for a typo.*

### 1.1.3 Coordinate transformations

It is intuitively clear that the sphere is intrinsically different from the plane. For example, two points on the Euclidean plane can be arbitrarily far, but on a sphere the maximal distance between two points is $2\pi R$. You can figure the metric on the sphere by embedding it in 3-dimensions:

$$z = R\cos\theta, \quad x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi,$$

with

$$0 \leq \theta \leq \pi, \quad -\pi < \phi \leq \pi$$

The metric is, by Pythagoras in Euclidean 3 space:

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2 = R^2(d\theta)^2 + R^2\sin^2\theta(d\phi)^2$$

The metric is close to Cartesian near the equator, $\theta = \pi/2$. But, since

$$\det g = R^4 \sin^2\theta$$

the metric has a coordinate singularity at $\theta = 0$ and $\theta = \pi$. Nothing bad happens at the poles, the sphere is still smooth there. It is only the coordinates which are messed up. This messing up is related to the fact that the poles belong to all time zones on earth.

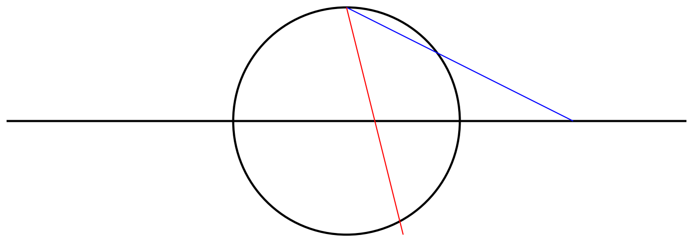

Figure 1.3: Projective representation of the sphere on the plane

**Remark 1.4.** *A choice of coordinates for the sphere, which has no finite coordinate singularity is the spherical projection, shown in the figure. The image of the south pole is at the origin, and the north pole at infinity.*

*The metric on the plane induced from the standard metric of the sphere is*

$$(ds)^2 = 4R^4 \frac{(dx)^2 + (dy)^2}{(R^2 + x^2 + y^2)^2}$$

*The metric tensor is diagonal*

$$g_{jk} = \delta_{jk} \frac{4R^4}{(R^2 + x^2 + y^2)^2} \tag{1.2}$$

*Note that the coordinates* $(x, y)$ *are dimensionless and* $\det g = 0$ *at aa single poiint: infinity.*

If $g$ is the metric tensor of some space (not necessarily Euclidean) in the coordinate $x$ and $\xi$ is different coordinate system of the same space, then the new metric $\gamma$ is

$$(d\ell)^2 = g_{ij} dx^i dx^j = \gamma_{ab} d\xi^a d\xi^b$$

where

$$\gamma = \Lambda^t g \Lambda, \quad \Lambda^i{}_a = \left( \frac{\partial x^i}{\partial \xi^a} \right) \tag{1.3}$$

### 1.1.4 Curvature: you may skip this

Given two metrics $g$ and $\gamma$ it is in general not a simple matter to decide if the two represent different coordinates of one space or the two spaces are intrinsically different, like a sphere and a plane. This problem was addressed by Gauss in the special case of two dimensional surface. Gauss showed that a necessary condition for the two surface to be the same is that the (Gaussian) curvatures coincide.

I shall not really give a proof of the fact that you can not get the metric of the sphere by making a coordinate transformation of the Euclidean metric, but instead hope that the following observation gives an idea how to do that: Since $g$

is a symmetric matrix, it can be diagonalized by an orthogonal transformation. So, at any given, fixed point $x$ there is a $\Lambda$ that diagonalizes $g$: You can choose coordinates so that at one point the metric looks like a Cartesian coordinates of Euclidean space. This is an expression of the fact that any (Riemannian) manifold is locally Euclidean. You can not make $g$ the identity everywhere unless the space is the Euclidean space and you choose Cartesian coordinates.

You can see this by counting the number of free parameters and the number of constraints in the Taylor expansions in a coordinate transformations near a point. You can choose coordinate transformations that make $g$ the identity and makes all its first derivative vanish, but can not make all the second derivatives zero. (The curvature is expressed in terms of second derivatives.)

## 1.2 Vectors: Contravariant components

The mother of all vectors is the velocity vector $\mathbf{v}$. We can associate with the velocity and infinitesimal displacement

$$d\mathbf{x} = \mathbf{v}dt \tag{1.4}$$

Consider such an infinitesimal displacement $d\mathbf{x}$ in the Euclidean plane. We want to represent the components of the vector in the (non-orthogonal) coordinate system shown in fig. 1.4:

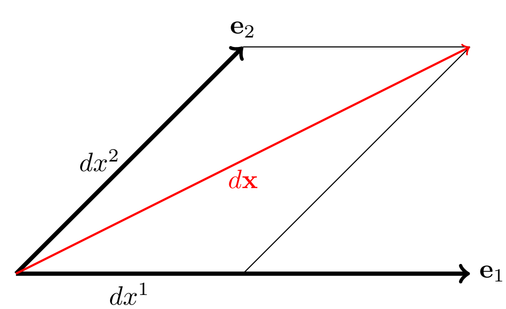


Figure 1.4: The contravariant components are the incremets $dx^j$ of the coordinates.

$$d\mathbf{x} = dx^1\mathbf{e}_1 + dx^2\mathbf{e}_2 \tag{1.5}$$

$\mathbf{e}_j$ are vectors pointing along the coordinate lines and $dx^j$ are called contravariant components. The vectors $\mathbf{e}_j$ give the directions. They are not, in general, unit vectors. Rather, they are related to the metric by

$$d\mathbf{x}\cdot d\mathbf{x} = \mathbf{e}_i\cdot\mathbf{e}_j\, dx^i dx^j = g_{jk}dx^j dx^k \tag{1.6}$$

$g$ is diagonal in orthogonal coordinate systems. The covariant basis vectors $\mathbf{e}_j$ are, in general, not unit vectors.

**Exercise 1.5.** *In a polar coordinate system the covariant basis vectors and the normalized unit vectors are related by*

$$\mathbf{e}_\rho = \hat{\boldsymbol{\rho}}, \quad \mathbf{e}_\theta = \rho\,\hat{\theta}$$

**Remark 1.6.** *Normalized unit vectors are defined* only *for orthogonal coordinate systems.*

Under a change of coordinates $x^j \mapsto \xi^j$, the transformation of the contravariant components $dx^j$ of a vector

$$(dx^1, dx^2) \mapsto (d\xi^1, \xi^2) \tag{1.7}$$

transform like the differentials of the coordinates, $dx^j$, i.e.

$$dx^j = \left(\frac{\partial x^j}{\partial \xi^a}\right) d\xi^a = \Lambda^j{}_a d\xi^a \Longleftrightarrow \underbrace{dx = \Lambda d\xi}_{Contravariant} \tag{1.8}$$

Think of the contravariant components as column vector and $\Lambda$ as a matrix.

This rule applies to the contravariant components of any vector, not just the infinitesimal displacement considered above.

**Exercise 1.7.** *The matrices*

$$\Lambda^j{}_a = \left(\frac{\partial x^j}{\partial \xi^a}\right), \quad (\Lambda^{-1})^b{}_k = \left(\frac{\partial \xi^b}{\partial x^k}\right)$$

*are inverses.*

**Example 1.8.** *The matrix $\Lambda$ that converts Cartesian components to Spherical components is*

$$\Lambda(C\text{-}to\text{-}S) = \frac{1}{r}\begin{pmatrix} r\cos(\phi)\sin(\theta) & r\sin(\theta)\sin(\phi) & r\cos(\theta) \\ \cos(\theta)\cos(\phi) & \cos(\theta)\sin(\phi) & -\sin(\theta) \\ -\csc(\theta)\sin(\phi) & \cos(\phi)\csc(\theta) & 0 \end{pmatrix} \tag{1.9}$$

### 1.2.1 Covariants components

The covariant components of the vector are given by drawing normals rather than parallels, as shown in fig. 1.5 We can write the vector in terms of the *dual* basis vectors $\mathbf{e}^j$ to the basis $\mathbf{e}_j$ defined by

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i{}^j \tag{1.10}$$

and $\delta$ is the Kronecker symbol. The vector $d\mathbf{x}$ can be represented in two different ways

$$d\mathbf{x} = dx^a \mathbf{e}_a = dx_a \mathbf{e}^a \tag{1.11}$$

Clearly

$$dx_j = d\mathbf{x} \cdot \mathbf{e_j} \tag{1.12}$$

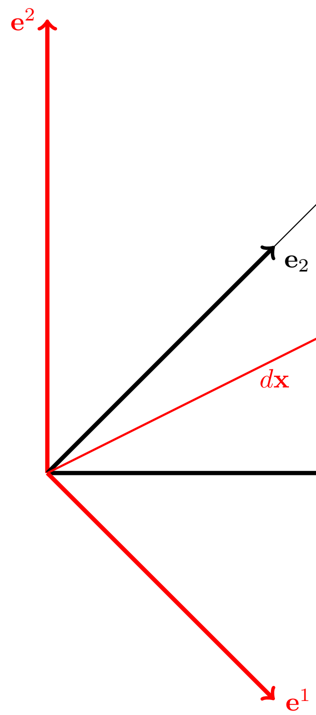


Figure 1.5: The covariant components are the incremets $dx_j$ of the orthogonal projections on the coordinates (schematic).

From the definitions of the metric tensor, and the notion of duality we get

$$dx_k = dx_a\, \mathbf{e}^a \cdot \mathbf{e}_k = dx^a\, \mathbf{e}_a \cdot \mathbf{e}_k = g_{ka} dx^a \tag{1.13}$$

*The metric tensor allows us to push indexes down.*

**Exercise 1.9.** *Show that*

$$\mathbf{e}^j = (\mathbf{e}^j \cdot \mathbf{e}^a)\, \mathbf{e}_a$$

The length of the vector $d\mathbf{x}$ is given by

$$d\mathbf{x} \cdot d\mathbf{x} = dx_a dx_b g^{ab} = dx^a dx^b g_{ab}, \quad g^{jk} = \mathbf{e}^j \cdot \mathbf{e}^k \tag{1.14}$$

Taking the scalar product of exercise 1.9 with $\mathbf{e}_k$ we conclude that $g_{jk}$ and $g^{jk}$ are inversely related

$$\delta^j_k \underbrace{=}_{duality} \mathbf{e}^j \cdot \mathbf{e}_k \underbrace{=}_{ex.10} (\mathbf{e}^j \cdot \mathbf{e}^a)(\mathbf{e}_a \cdot \mathbf{e}_k) = g^{ja} g_{ak} \underbrace{=}_{index\ gym} g^j{}_k$$

$g^{jk}$ raises indexes since

$$g^{ja} dx_a = g^{ja} g_{ab} dx^b = \delta^j_b dx^b = dx^j \tag{1.15}$$

The transformation rules for the covariant components of the position vector follow:

$$\begin{aligned} d\xi_a &= \gamma_{ab} d\xi^b \\ &= \gamma_{ab}(\Lambda^{-1}dx)^b \\ &= \Lambda^i{}_a \Lambda^j{}_b g_{ij} (\Lambda^{-1})^b{}_k dx^k \\ &= \Lambda^i{}_a \Lambda^j{}_b (\Lambda^{-1})^b{}_k g_{ij}\, dx^k \\ &= \Lambda^i{}_a (\Lambda\Lambda^{-1})^j{}_k g_{ij}\, dx^k \\ &= \Lambda^i{}_a g_{ij}\, dx^j \\ &= \Lambda^i{}_a dx_i \\ &= dx_i \Lambda^i{}_a \end{aligned}$$

Contrast this with the transformation rule of the contravariant component

$$dx^a = \Lambda^a{}_i d\xi^i$$

Since all these indices can make one dizzy, let me write the two rules in a way that make the comparison simple. If you write the covariant components as *a row* vector and contravariant as a columns, the transformations can be written in matrix and vector notation as

$$\underbrace{d\boldsymbol{\xi}^t = d\mathbf{x}^t \Lambda}_{\text{covariant components}} \quad , \qquad \underbrace{d\mathbf{x} = \Lambda d\boldsymbol{\xi}}_{\text{contravariant components}} \tag{1.16}$$

The comparison between the rules of transformations of becomes even more transparent when we write both as column vectors:

$$\underbrace{d\boldsymbol{\xi} = \Lambda^t \mathbf{x}}_{\text{covariant components}} \quad , \qquad \underbrace{d\boldsymbol{\xi} = \Lambda^{-1} d\mathbf{x}}_{\text{contravariant components}} \tag{1.17}$$

The covariant components transform by $\Lambda^t$ and the contravariant by $\Lambda^{-1}$. In general, these two rules are different, but they coincide in the special case that $\Lambda$ is an orthogonal transformation, where $\Lambda^t = \Lambda^{-1}$. This is the reason why we need no distinguish covariant from contravariant components in (orthogonal) Cartesian coordinates.

Now, by decree, the same rules of transformations hold for any vector, not just the coordinates. Hence, if $v$ is any vector in the $x$ coordinates, and $\nu$ the corresponding vector in the $\xi$ coordinates the components are related by

$$\nu_k = u_j \Lambda^j{}_k, \quad u^k = \Lambda^k{}_j \nu^j \tag{1.18}$$

Note the interchange $u \leftrightarrow \nu$ and since the summation indexes $j$ are adjacent $u_j$ should be considered as a row vector and $\nu^j$ as column.

### 1.2.2 Contraction makes scalars

It follows from the first eqaution in 1.16 and the second equation in 1.17 that the contraction $u_j v^j = \mathbf{u}^t \mathbf{v}$ is coordinate independent. Indeed, if we denote $u'$ and $v'$ the corresponding transformed coordinates then

$$(u')_j (v')^j = \mathbf{u'}^t \mathbf{v'} = \mathbf{u}^t \Lambda \Lambda^{-1} \mathbf{v} = \mathbf{u}^t \mathbf{v} = u_j v^j \tag{1.19}$$

### 1.2.3 Orthogonal coordinates

Many of the standard curvilinear coordinate systems one encounters in practice are orthogonal. In this case, the metric $g$ is a diagonal matrix. Orthogonal coordinates admit three types of components: The usual covariant and contravariant components and the "normalized" components. All three are given by

$$v^j \mathbf{e}_j = v_j \mathbf{e}^j = v_{\hat{j}} \mathbf{n}_j$$

with $\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij} = g_i \delta_{ij}$, $\mathbf{e}^i \cdot \mathbf{e}^j = g^{ij} = (g_i)^{-1} \delta_{ij}$ and $\mathbf{n}_j \cdot \mathbf{n}_j = \delta_{ij}$.

**Example 1.10** (Polar coordinates)**.** *In polar coordinates*

$$\mathbf{v} = \underbrace{v^\rho \mathbf{e}_\rho + v^\theta \mathbf{e}_\theta}_{contravariant\ components} = \underbrace{v_\rho \mathbf{e}^\rho + v_\theta \mathbf{e}^\theta}_{covariant\ components} = \underbrace{v_{\hat{\rho}} \hat{\boldsymbol{\rho}} + v_{\hat{\theta}} \hat{\boldsymbol{\theta}}}_{normalized} \tag{1.20}$$

*The local frames have basis vectors:*

$$\begin{aligned} \mathbf{e}_\rho \cdot \mathbf{e}_\rho = 1, \quad \mathbf{e}_\theta \cdot \mathbf{e}_\theta = \rho^2, \quad \mathbf{e}_\rho \cdot \mathbf{e}_\theta = 0 \\ \mathbf{e}^\rho \cdot \mathbf{e}^\rho = 1, \quad \mathbf{e}^\theta \cdot \mathbf{e}^\theta = \rho^{-2}, \quad \mathbf{e}^\rho \cdot \mathbf{e}^\theta = 0 \\ \hat{\boldsymbol{\rho}} \cdot \hat{\boldsymbol{\rho}} = 1, \quad \hat{\theta} \cdot \hat{\theta} = 1, \quad \hat{\boldsymbol{\rho}} \cdot \hat{\theta} = 0 \end{aligned}$$

**Example 1.11** (Mechanical model)**.** *A particle of unit mass moves on a ring of fixed radius, $r$. Its orbit in polar coordinates is $(r, \theta(t))$. Its velocity vector is:*

$$\mathbf{v} = \underbrace{(r\dot{\theta})}_{velocity} \hat{\boldsymbol{\theta}} = \underbrace{\dot{\theta}}_{angular\ velocity} \mathbf{e}_\theta = \underbrace{(r^2 \dot{\theta})}_{\text{``}angular\ momentum\text{''}} \mathbf{e}^\theta \tag{1.21}$$

## 1.3 Scalars, vectors, tensors

The charge of a particle, its mass, or the length of a vector are all scalars: You do not need to decide on coordinates to give a numerical value of scalars. If you do use coordinates, the result should be independent of the choice of coordinates. This is the defining property of scalars.

Vectors are geometric objects and as such do not rely of a coordinate system either. But, their representation by their components, covariant or contravariant, depend on the choice of coordinate system. Vectors are also called rank one tensors because their components have a single index. The rule of transformation under coordinate change, as we have seen is:

$$\nu_k = u_j \Lambda^j{}_k, \quad u^k = \Lambda^k{}_j \nu^j \tag{1.22}$$

Tensors are multi-index objects that transform as the product of vectors. The number of indices of the tensor is called each rank. Each index transforms according to whether it is up or down.

The metric tensor is an example of a symmetric second rank tensor. In particular, the metric of a coordinate transformation of the Euclidean plane discussed in section 1.1.1, follows this rule:

$$g_{jk} = (\Lambda^t \Lambda)_{jk} = (\Lambda^t \mathbb{1} \Lambda)_{jk} = (\Lambda^t)_j{}^a \delta_{ab} \Lambda^b{}_k = \delta_{ab} \Lambda^a{}_j \Lambda^b{}_k \tag{1.23}$$

**Exercise 1.12.** *The identity, $\delta^j{}_k$, a mixed second rank tensor, is invariant under coordinate transformations.*

### 1.3.1 Symmetric and anti-symmetric tensors

Coordinate transformations respect the symmetry of tensors: If $T$ is symmetric (anti-symmetric), i.e. $T_{jk} = \pm T_{kj}$, so is $(T')$.

**Exercise 1.13.** *Show that if $T_{jk}$ is symmetric (anti-smmetric) so is $T^{jk}$ but in general $T_j{}^k \neq T_k{}^j$. (One finds instead $T_j{}^k = T^k{}_j$).*

### 1.3.2 Densities and Weights

Scalars and tensors are basic objects in the equation of nature. But it turns out that there there are subtle features that force us to distinguish tensors from objects that are almost tensors. For example, $\det g$ has no indices, but is not quite a scalar since by Eq. 1.3

$$\det \gamma = \big(\det \Lambda\big)^2 \det g$$

Indeed, $\det g = 1$ in Cartesian coordinates; it is invariant under orthogonal transformations. But, it is not a scalar under change of scale of the coordinates: One says that $\det g$ is a scalar with density $-2$.

### 1.3.3 Volume

$\det g$ enters into the volume element $dV$. For the sake of simplicity, let us consider the case of two dimensions where volume is area. Recall that the (signed) area of the parallelogram associated with the two vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \mathbf{u} \wedge \mathbf{v}$$

and $\wedge$ is a generalization of $\times$ to any dimension. We have for the (signed) area

$$dV = dx^1 \mathbf{e}_1 \wedge dx^2 \mathbf{e}_2 = \; \mathbf{e}_1 \wedge \mathbf{e}_2 \; dx^1 dx^2 = \mathbf{z} |\mathbf{e}_1| |\mathbf{e}_2| \sin\theta \; dx^1 dx^2$$

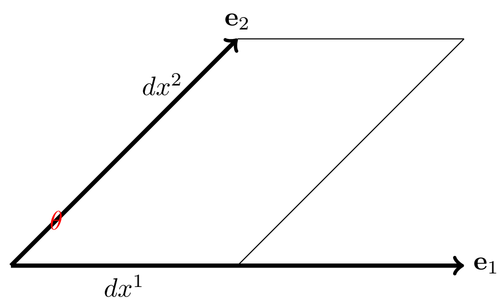


Figure 1.6: The area of a parallelogram

where **z** denote the notion of oriented unit area. Observing that

$$g = \begin{pmatrix} |\mathbf{e}_1|^2 & |\mathbf{e}_1||\mathbf{e}_2|\cos\theta \\ |\mathbf{e}_1||\mathbf{e}_2|\cos\theta & |\mathbf{e}_2|^2 \end{pmatrix}$$

we conclude that

$$dV = \mathbf{z}\sqrt{\det g}\, dx^1 dx^2$$

Objects with such a rule of transformation are called weights. $\det g$ has weight $-2$.

**Exercise 1.14** (Spherical coordinates)**.** *Let $(x, y, z)$ be Cartesian coordinates off Eucliden space with metric $d\ell^2 = (dx)^2 + (dy)^2 + (dz)^2$. The standard spherical coordinates*

$$z = r\cos\theta, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi$$

*have the metric tensor*

$$d\ell^2 = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta\,(d\phi)^2$$

*and that the volume element is*

$$dV = \sqrt{\det g}\; dr\, d\theta\, \mathrm{d}\phi = r^2\sin\theta\; dr\, d\theta\, \mathrm{d}\phi = -r^2\; dr\, d\cos\theta\, \mathrm{d}\phi$$

### 1.3.4 Levi-Civita tensor and symbol

We can always define a tensor by giving its numerical components in a particualr coordinate system, and then declare that it is given in any other coordinate system by the rules of tensor transformations.

Pick a Cartesian coordinates in two dimensional Eucliden space and define the Levi-Civita tensor by setting

$$\varepsilon^{xx} = \varepsilon^{yy} = 0, \quad \underbrace{\varepsilon^{xy}}_{even} = \underbrace{-\varepsilon^{yx}}_{odd} = 1$$

Since the Cartesian metric is the identity we also have

$$\boldsymbol{\varepsilon}_{xx} = \boldsymbol{\varepsilon}_{yy} = 0, \quad \underbrace{\boldsymbol{\varepsilon}_{xy}}_{even} = \underbrace{-\boldsymbol{\varepsilon}_{yx}}_{odd} = 1$$

In other coordinates the transformation rule of tensors imply

$$\boldsymbol{\varepsilon}'_{12} = \boldsymbol{\varepsilon}_{ab}\Lambda^a{}_1\Lambda^b{}_2 = \det\Lambda, \quad \boldsymbol{\varepsilon}'_{11} = \boldsymbol{\varepsilon}_{ab}\Lambda^a{}_1\Lambda^b{}_1 = 0, \quad a, b, \in x, y$$

It is always anti-symmetric, but its numerical value is not, in general, the identity except if $\Lambda$ is a rotation so $\det\Lambda = 1$. For example, in polar coordinates, the Levi-Civita tensor would be

$$\boldsymbol{\varepsilon}_{\rho\theta} = -\boldsymbol{\varepsilon}_{\theta,\rho} = \rho, \quad \boldsymbol{\varepsilon}_{\rho\rho} = \boldsymbol{\varepsilon}_{\theta\theta} = 0$$

and

$$\boldsymbol{\varepsilon}^{\rho\rho} = \boldsymbol{\varepsilon}_{\theta,\theta} = 0, \quad \boldsymbol{\varepsilon}^{\rho\theta} = -\boldsymbol{\varepsilon}^{\theta\rho} = \frac{1}{\rho}$$

In any dimension, the Levi-Civita *symbol* is defined as the highest rank of completely anti-symmetric tensor with components $\pm1$ and 0. In particular in three dimensions

$$\underbrace{\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312}}_{even\ permutations} = \underbrace{-\varepsilon^{321} = -\varepsilon^{213} = -\varepsilon^{132}}_{odd\ permutations} = 1 \tag{1.24}$$

The Levi-Civita symbol is a tensor with density. The relation between the Levi-Civita tensor (bold) and symbol is,

$$\underbrace{\boldsymbol{\varepsilon}_{i\ldots j}}_{tensor} = \sqrt{\det g}\ \underbrace{\varepsilon_{i\ldots j}}_{symbol}, \quad \underbrace{\boldsymbol{\varepsilon}^{i\ldots j}}_{tensor} = \frac{1}{\sqrt{\det g}}\ \underbrace{\varepsilon^{i\ldots j}}_{symbol} \tag{1.25}$$

**Remark 1.15.** *An orthogonal coordinate system is right handed if*

$$\varepsilon^{1\ldots d} = 1$$

**Remark 1.16.** *In even dimensions cyclic permutations are odd, while in odd dimensions cyclic permutations are even.*

**Exercise 1.17.** *Show that in $n$ dimensions*

$$\varepsilon^{ij\cdots}\varepsilon_{ij\ldots} = n!$$

**Exercise 1.18.** *Show that (in 3 dimensions)*

$$\varepsilon^{ijk}\varepsilon_{iab} = \delta^j_a\delta^k_b - \delta^j_b\delta^k_a, \quad \varepsilon^{ijk}\varepsilon_{ijb} = 2\delta^k_b$$

## 1.4 Tensors and pseudo-tensors

Inversion of Cartesian coordinates is associated with $\Lambda = -\mathbb{1}$. The Cartesian components of ordinary vectors flip signs under inversion: $(v')^j = -v^j$. (The vector still points in the same direction.)

However, we sometimes encounter in physics vectors that do not flip sign. These are pseudo vectors.

The angular momentum is an example:

$$\mathbf{L}^i = (\mathbf{x} \times \mathbf{p})^i = \varepsilon^{ijk} x_j p_k \Longrightarrow L^j = L'^j$$

with $\varepsilon$ the Levi-Civita symbol.

## 1.5 Isometries of Euclidean space

Euclidean space looks the same no matter where you are or how you are oriented: It is homogeneous and isotropic. These symmetries reflect invariance properties of the metric tensor of Euclidean space under suitable coordinate transformations. In Cartesian coordinates $x^j$ a shift

$$(x')^j = x^j + a^j \Longrightarrow \Lambda = \mathbb{1} \Longrightarrow g' = g = \mathbb{1},$$

leaves $g$ invariant, reflecting the homogeneity of Euclidean space. The components of vectors are invariant under coordinate shift: $(v')^j = v^j$. This is what is meant by saying that vectors in Cartesian system do not have a location.

Rotation is a linear transformation of the Cartesian coordinates that keeps the origin fixed

$$(x')^j = \Lambda^j{}_a x^a \tag{1.26}$$

and leaves the metric invariant. By definition, coordinates are Cartesian if $g = \mathbb{1}$. And indeed, a rotation of Cartesian coordinates are rotates Cartesian coordinates:

$$\mathbb{1} = (g') = \Lambda^t g \Lambda = \Lambda^t \mathbb{1} \Lambda = \Lambda^t \Lambda$$

This says that $\Lambda^t$ is the inverse of $\Lambda$:

$$\Lambda^t \Lambda = \mathbb{1}$$

which is the standard definition of orthogonal transformation. It follows that

$$\det \Lambda^2 = 1 \Longrightarrow \det \Lambda = \pm 1$$

Orthogonal transformations are associated with two types of symmetries of the Euclidean space: When $\det \Lambda = 1$ they represent rotations. When $\det \Lambda = -1$ they represent reflections.

**Example 1.19** (Rotations)**.** *In three dimensions, rotation by $\theta$ about the $x^3$ axis is given by*

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

*The inverse is the transpose*

$$R(-\theta) = R^t(\theta) \tag{1.27}$$

**Example 1.20.** *In a two dimensional Euclidean space there are two second rank tensors that are invariant under rotations (up to multiplication by a scalar): The identity and Levi-Civita*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.28}$$

## 1.6 Tensorial equations are coordinate free

The nice thing about tensor equations is that once

$$T^{jk\dots} = 0$$

holds in one (fixed) coordinate system, it hold in any other coordinate system. For example, Newton's equation

$$f^j = ma^j$$

is a tensor equation relating force and acceleration ($m$ is a scalar). If it holds in one coordinate system it hold in any other.

## 1.7 Differential operators

The equations of motions of fields are partial differential equations. This forces us to mind how differential operators behave under change of coordinates. The general case is complicated and one needs to introduce the notion of covaraint derivatives. A simplification however occurs for the three differential operators we need for Maxwell's equations: grad, div and curl.

### 1.7.1 Grad

The chain rule

$$\frac{\partial}{\partial \xi^j} = \frac{\partial x^k}{\partial \xi^j}\frac{\partial}{\partial x^k} = \Lambda^k{}_j \frac{\partial}{\partial x^k}$$

says that partial derivatives of scalar functions behave like *covariant* components of a vector. That is, if $\phi(x)$ is a scalar field then $\nabla\phi$ give the components of a covariant vector field.

**Exercise 1.21.** *Show that* $\nabla\phi$ *in spherical coordinates is*

$$\begin{aligned}\nabla\phi &= (\partial_r\phi)\,\mathbf{e}^r + (\partial_\theta\phi)\,\mathbf{e}^\theta + (\partial_\varphi\phi)\,\mathbf{e}^\varphi \\ &= (\partial_r\phi)\,\hat{\mathbf{r}} + \frac{\partial_\theta\phi}{r}\,\hat{\theta} + \frac{\partial_\varphi\phi}{r\sin\theta}\,\hat{\varphi}\end{aligned}$$

The covariant components lead to simple formulas while the normalized components are a mess.

### 1.7.2 Div

In any coordinate system

$$\nabla\cdot\mathbf{E} = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}E^j) \tag{1.29}$$

The formulas clearly hold in Cartesian coordinates where $g$ is the identity.

The defining property of the divergence is Gauss law

$$\int_{Volume} dV(\nabla\cdot E) = \int_{surface} dS\cdot E$$

Consider a smal cube in the coordinates $dx^j$. The putative expression for div indeed satisfies Gauss law

$$\begin{aligned}dV(\nabla\cdot E) &= \sqrt{g}dx^1dx^2dx^3(\nabla\cdot E) \\ &= dx^2dx^3\sqrt{g}\,E^1\Big|_i^f + \dots \\ &= dx^2dx^3\sqrt{g}\mathbf{e}^1\cdot E^1\mathbf{e}_1\Big|_i^f + \dots \\ &= \underbrace{\sqrt{g}\,dx^2dx^3\mathbf{e}^1}_{dS}\cdot\mathbf{E}\Big|_i^f + \dots \\ &= d\mathbf{S}\cdot\mathbf{E}\end{aligned}$$

**Example 1.22.** *In spherical coordinates div is*

$$\begin{aligned}\nabla\cdot E &= \frac{1}{r^2\sin\theta}\left(\partial_r(r^2\sin\theta E^r) + \partial_\theta(r^2\sin\theta E^\theta) + \partial_\phi(r^2\sin\theta E^\phi)\right) \\ &= \frac{1}{r^2}\partial_r(r^2E^r) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta E^\theta) + \partial_\phi(E^\phi) \\ &= \frac{1}{r^2}\partial_r(r^2E_{\hat{r}}) + \frac{1}{r\sin\theta}\left(\partial_\theta(\sin\theta E_{\hat{\theta}}) + \partial_\phi(E_{\hat{\phi}})\right)\end{aligned}$$

*The last line is in terms of the normalized coordinates.*

### 1.7.3 Curl

An important differential operator we shall need to discuss is the curl:

$$(\nabla \times E)^i = \frac{\varepsilon^{ijk}}{\sqrt{g}} \partial_j E_k = \frac{\varepsilon^{ijk}}{2\sqrt{g}} \underbrace{(\partial_j E_k - \partial_k E_j)}_{\text{anti-symmetric tensor}} \tag{1.30}$$

The term on the right is the contraction of the Levi-Civita tensor with an anti-symmetric tensors so the object is a bona-fide vector.

Eq. 1.30 is evidently the standard definition in Cartesian coordinates. To see why it is true in general we take Stokes law as defining property of the curl:

$$\int dS \cdot (\nabla \times \mathbf{E}) = \int d\ell \cdot \mathbf{E}$$

The putative formula for curl indeed gives Stokes for $dS$ a small square $dx^1 \times dx^2$.

$$\begin{aligned} dS \cdot (\nabla \times E) &= dS_3 \, (\nabla \times E)^3 \\ &= \left(\sqrt{g} dx^1 dx^2\right) \left( \frac{\varepsilon^{3ij}}{\sqrt{g}} \partial_i E_j \right) \\ &= dx^1 dx^2 (\partial_1 E_2 - \partial_2 E_1) \\ &= dx^2 E_2 \Big|_i^f - \ldots \\ &= (dx^2 \mathbf{e}_2) \cdot (E_2 \mathbf{e}^2) \Big|_i^f - \ldots \\ &= (dx^2 \mathbf{e}_2) \cdot \mathbf{E} \Big|_i^f - \ldots \\ &= d\ell \cdot \mathbf{E} \end{aligned}$$

**Example 1.23.** *The $\phi$ components of curl in spherical coordinates* [1] *(recall Remark1.15 ) is :*

$$(\nabla \times \mathbf{E})^\varphi = \frac{1}{r^2 \sin\theta} \left(\partial_\theta E_r - \partial_r E_\theta\right)$$

*and in normalized components*

$$\begin{aligned} (\nabla \times \mathbf{E})_{\hat{\varphi}} &= \frac{1}{r} \left(\partial_\theta E_r - \partial_r E_\theta\right) \\ &= \frac{1}{r} \left(\partial_\theta E_{\hat{r}} - \partial_r (r E_{\hat{\theta}})\right) \end{aligned}$$

**Exercise 1.24.** *Compute the $(\nabla \times \mathbf{E})^r$ and $(\nabla \times \mathbf{E})_{\hat{r}}$.*

**Exercise 1.25** (Vector identities)**.** *Show the vector identities*

$$\begin{aligned} \nabla \times (\nabla \phi) &= 0 \\ \nabla \cdot (\nabla \times \mathbf{E}) &= 0 \end{aligned}$$

[1] Note that Wolfram Mathematica notation compares with mine by interchanging $\varphi \leftrightarrow \theta$

### 1.7.4 Laplacian

The Laplacian of a scalar function is defined by

$$\Delta\phi = \nabla \cdot \nabla\phi = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k\phi) = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}\,\partial^j\phi)$$

and for a vector field by

$$\nabla \times (\nabla \times \mathbf{E}) = -\Delta\mathbf{E} + \nabla(\nabla \cdot \mathbf{E})$$

**Example 1.26.** $\Delta\phi$ *in spherical coordinates:*

$$\begin{aligned}\Delta\phi &= \frac{1}{r^2\sin\theta}\left(\partial_r(r^2\sin\theta\partial_r\phi) + \partial_\theta\left(r^2\sin\theta\,\frac{\partial_\theta\phi}{r^2}\right) + \partial_\varphi\left(r^2\sin\theta\frac{1}{r^2\sin^2\theta}\partial_\varphi\phi\right)\right)\\ &= \frac{1}{r^2}\partial_r(r^2\partial_r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\phi\right) + \frac{1}{r^2\sin^2\theta}\partial_{\varphi\varphi}\phi\end{aligned}$$

## 1.8 Bibliography


- S. Weinberg, Gravitation and Cosmology, gives all a physicist needs to know about tensors.
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# Chapter 2

# Review of special relativity: Minkowski space-time

Minkowski space-time is a good approximation of our physical space-time. (This would not be the case if we lived near a black hole.) It gives the geometric setting of special relativity and encapsulates the fundamental observations that: *The velocity of light $c$ is finite and has the same value in all inertial frames.*

## 2.1 The principle of relativity

Physicists unlike, say, lawyers, do not need to replace their textbooks when they relocate. Your physics library would still be useful even if you relocated to a different earth like planet is a different galaxy. You do not need to make an adjustment for the relative motion between earth and your new home and or how the new planet is oriented relative to earth.

Empty space, far from any planet or star, has no distinguished inertial frame and no distinguished origin or orientation. The speed of light $c$ is a scalar, a constant of nature, which takes the same value in all inertial frames. This counter-intuitive property of light was established around 1887 in experiments of Michelson and Morley. It conflicts with our common intuition about adding velocities much smaller than $c$.

**Remark 2.1** ($c$ is large)**.** *Practical units, like MKS, are chosen to be of $O(1)$ on human scale. The period of a pendulum of a meter length (about the length of a leg) is about 2 [sec] and the second is about a heart beat. However, a light second is about the distance to the moon. On human scale $c \approx 3 \times 10^8$ $[m/s]$ is essentially infinite. It is likely that ancient natural scientist entertained the thought that $c$ is finite rather than infinite. (Maimonides cautions against logical pitfalls resulting from the use of infinities.) But as $c$ is so large it is difficult to devise an elementary experiment to measure it. The first to estimate $c$ from irregularities in motion of the moons of Jupiter was the Danish astronomer*

*Rømer (1644-1710). He essentially used the Doppler effect (two hundred years before Doppler) to measure the ratio $v/c$ where $v$ is the velocity of earth around the sun.*

## 2.2 Space-time

Space-time is the stage on which events happen. An event, like my typing this text, is something that happens in space and time and is labeled by 4-coordinates $(t, \mathbf{x})$. It is convenient to measure space and time in the same units, e.g. measure distances in light second, or alternatively, measure time in light-meter. With the latter choice we write

$$(ct, x^1, x^2, x^3) = \{x^\mu\}, \quad \mu = 0, \dots 3$$

#### Index convention

Greek indices $\mu, \nu$ run on $0, 1, 2, 3$. Roman indices $j, k$ on $1, 2, 3$

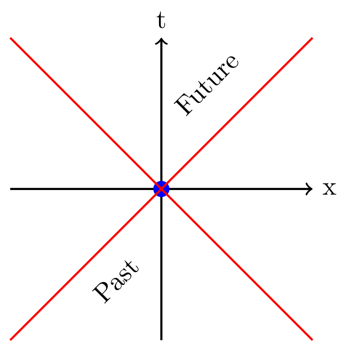


Figure 2.1: Space-time: The red lines represent the light cones. The blue dot at the origin is the event where light was emitted (and absorbed).

The (Cartesian) metric of Minkowsky space is commonly denoted by $\eta$:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

It allows to associate a scalar with a vector. In the case of a vector $dx^j$ relating two nearby events in space-time the scalar is called *interval*

$$(d\ell)^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(cd\tau)^2 \tag{2.2}$$

Note that $(d\ell)^2$ is indefinite. When $(d\ell)^2 > 0$ it has a spatial character and has a real root measured in [cm]. When $(d\ell)^2 < 0$ it has time-like character and it is then convenient to consider instead the real root of $(d\tau)^2 > 0$ which is measured in $[sec]$.

- A vector $v^\mu$ is called space like if $v_\mu v^\mu > 0$;
- $v^\mu$ is called time-like if $v_\mu v^\mu < 0$
- $v^\mu$ is light-like if $v_\mu v^\mu = 0$
- The 1-dimensional line $\mathbf{x} = 0$ is the time axis, and the clock attached to the origin measures time $\tau$ related to the interval by $(c\tau)^2 = -(d\ell)^2 \geq 0$.
- The 3-dimensional portion of space time

  $$\eta_{\mu\nu} x^\mu x^\nu = 0$$

  is a 3-dimensional light cone.

**Example 2.2.** *In spherical coordinates* $(ct, r, \theta, \phi)$ *Minkoski metric is*

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{2.3}$$

### 2.2.1 Events, world line and proper-time

An event is an objective happening and does not depend on the choice of coordinates. However, its representation in terms of coordinates $x = (x^0, x^1, x^2, x^3)$ depends of the choice of coordinates.

A line in space-time is called a world line. For example, the collection of events associated with a clock (moving at subluminal speed) gives the world line of the clock.

Consider a curve in space-time parametrized by Lab-time

$$\mathbf{X}(t) = (ct, \mathbf{x}(t)), \quad -\infty < t < \infty$$

The velocity is a 4-vector which is tangent to the world line

$$\mathbf{V} = \dot{\mathbf{x}}(t) = (c, \mathbf{v}(t)), \quad \mathbf{v} = \dot{\mathbf{x}}$$

A massive particle that does not move faster than $c$ has time-like $\mathbf{V}$ at all times.

The Minkowaski length of a world line is a scalar. Its physical meaning is the time measured by a standard clock moving along the path

$$\tau = \frac{1}{c}\int_0^t \sqrt{-(d\ell)^2} = \int_0^t \frac{dt'}{\gamma(t')} < t, \quad \gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}} \tag{2.4}$$

It is a common, and depressing, fact that when you meet your high school buddies after a long time you note how everybody else aged. This is psychology, and I have nothing to say about it. There is an analog objective property of time: Your clock is slower than the lab clock and traveling keeps you young.

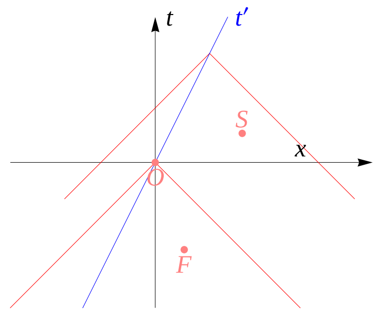


Figure 2.2: The backward light cone of a given point is the collection of the events in your past that you can observe. An inertial observer that lives long enough eventually sees all the events in Minkowski space-time.

**Exercise 2.3** (Measuring intervals with a clock. Skip on first reading)**.** *To measure space-like intervals you would normally use meter sticks. Wigner found a clever trick to measure space like intervals using a clock and light signals. This is illustrated in Fig. 2.3 and is expressed by the formula*

$$(x_O - x_S)^\mu (x_O - x_S)_\mu = ab$$

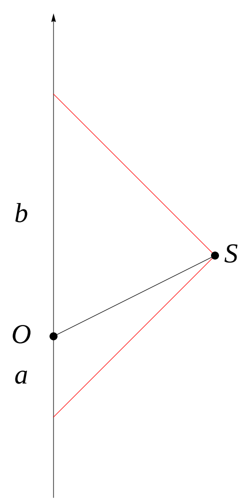


Figure 2.3: Wigner method of measuring space-like interval $OS$ with a clock. The interval between the two space like events $O$ and $S$ is related to the clock readings $a, b$. The red lines in the figure are light-like.

## 2.3 Simultaneity

Two events $\mathbf{X}_1$ and $\mathbf{X}_2$ occur simultaneously in the lab if

$$\Delta\mathbf{X} = \mathbf{X}_1 - \mathbf{X}_2 = (0, \mathbf{x}_1 - \mathbf{x}_2)$$

$\Delta\mathbf{X}$ is Minkowski orthogonal to the lab-time vector $\hat{\mathbf{T}} = (1, 0, 0, 0)$

$$(\Delta X)^\mu \hat{T}_\mu = 0$$

Now consider the notion of simultaneity in an inertial frame moving at uniform speed $\mathbf{v}$ with respect to the Lab. The time-like direction is

$$(c, \mathbf{v})$$

and the corresponding unit vector (in Minkowski metric) is

$$\hat{\mathbf{T}}' = \gamma \left(1, \frac{\mathbf{v}}{c}\right)$$

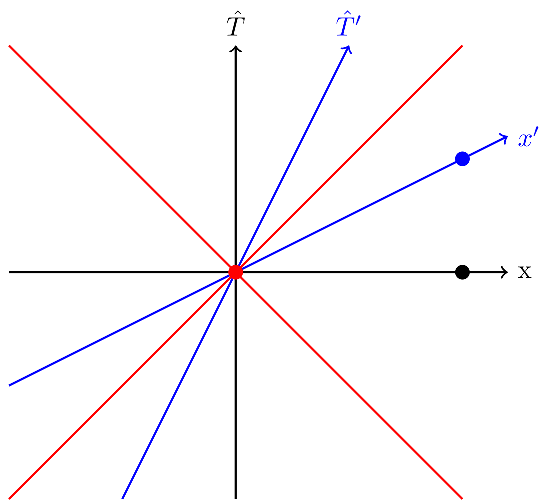


Figure 2.4: The vector $\hat{T}$ is the time direction in the lab and the black dot is simultaneous with the red dot in the lab. The vector $\hat{T}'$ is the time direction in a moving inertial frame and the blue dot is simultaneous with the red dot in this frame.

Two events are simultaneous in the moving frame if the 4-vector of their difference $\Delta X$ is Minkowski orthogonal to $\hat{T}'$, i.e.

$$(\Delta X)^\mu \hat{T}'_\mu = 0$$

In particular, the events simultaneous with the event at the origin $(0,0,0,0)$ are given by the points $(x^0, \mathbf{x})$ that satisfy

$$cx^0 - \mathbf{v} \cdot \mathbf{x} = 0$$

The two straight lines marked $x$ and $x'$ in Fig. 2.4 show the events simultaneous with the origin in the two frames.

The notion of now in the lab frame and the notion of now in the inertial frame of the clock are incompatible. Now is a well defined concept at a point in space-time.

### 2.3.1 Time dilation and length contraction

Eq. 2.4 makes it clear that proper-time is the shorter than lab time. In contrast, proper length (measured in the frame of the moving object) is the longer length. This seems confusing. The figures below, and their captions, try to explain time dilation and length contraction with pictures.

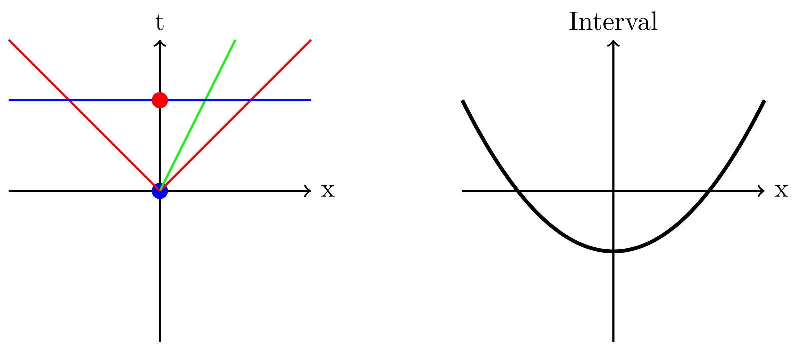


Figure 2.5: The Minkowski “distance” $(d\ell)^2$ between the origin and an event $t = 1$ in the lab, the blue line, takes negative values inside the light cone and positive values outside. The minimum (negative value) is taken at the red dot. This corresponds to the proper time of a clock which is at rest at the lab. A moving clock, as shown by the green line, registers $cd\tau = |d\ell|$. The intersection with the blue line will give a smaller value for the proper time of the moving clock.

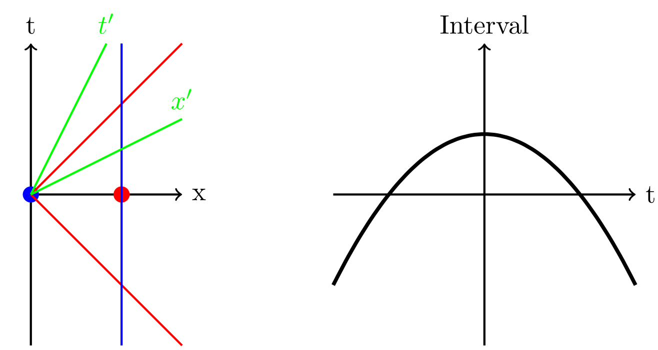


Figure 2.6: The blue and red dots are the ends of a ruler of length 1 at rest in the lab. The straight blue line is the world line of the right end. The Minkowski distance between the origin and the blue line takes its maximum at the red dot. The green frame is Lorentz frame where the rod is moving. $x'$ is the time 0 slice in the moving frame. It intersects the blue line at a point whose Minkowski distance is smaller than one. The proper length is largest in the rest frame.

**Exercise 2.4.** *Suppose that you have a factory at the origin that makes identical clocks. Explain how you can distribute the clocks while keeping them (approximately) synchronized in the Lab.* (Hint: What happens to time delay if you half the speed and double the travel time?)

**Remark 2.5.** *In the 1970's, when Paul Krugman, A Nobel Laureate in economics, was a young assistant professor he wrote an amusing article about the consequences of time-dilation in economics. It can be found here and is worth reading.*

### 2.3.2 Light-cone coordinates

Light cone coordinates in Minkowski space are

$$\sqrt{2}u = x - ct, \quad \sqrt{2}v = x + ct, \quad y = y, \quad z = z$$

**Exercise 2.6** (Metric in light cone coordinates)**.** *Show that the Minkowski metric tensor in light-cone ordinates is*

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

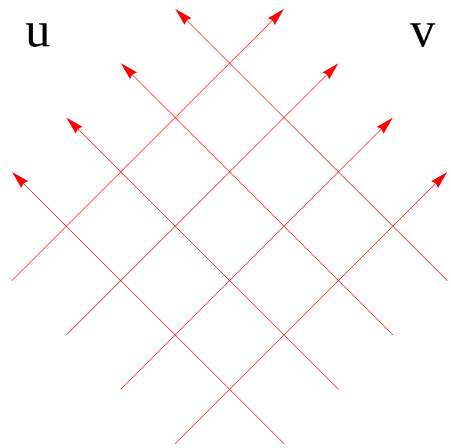


Figure 2.7: Light-cone coordinates. This is not a Cartesian frame since $\mathbf{u}\cdot\mathbf{v} = 1$.

## 2.4 4-Velocity

The proper time $d\tau$ is a Lorentz scalar. It is non-zero and real for a clock that travels subluminally. We can then define the velocity as a 4-vector

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{2.5}$$

The length of $u$ is always $-c^2$ , essentially by the definition of the interval, Eq. 2.2,

$$u_\mu u^\mu = \frac{dx_\mu dx^\mu}{(d\tau)^2} = \frac{(d\ell)^2}{(d\tau)^2} = -c^2 \tag{2.6}$$

*The 4-velocity is therefore a time-like vector* which lies in the forward light cone. It is related to the usual velocity by

$$(c, \mathbf{v}) = \frac{dx^\mu}{dt} = \frac{dx^\mu}{d\tau}\frac{d\tau}{dt} = \frac{u^\mu}{\gamma} \tag{2.7}$$

**Remark 2.7** (Newtonian velocities)**.** *The components of the 4-velocity transform like a contravariant 4-vector under Lorentz transformations. In contrast, the Newtonian velocity has complicated and ugly transformation properties. This is the result of the fact that for the Newtonian velocity both the numerator and the denominator are components of a vector.*

Think of a world line $x^\mu(\tau)$ as parametrized by proper time. Since the 4-velocity is normalized and time-like, we can always write it as

$$u = \dot{x} = c\,(\cosh\phi, \mathbf{n}\,\sinh\phi), \quad \mathbf{n}\cdot\mathbf{n} = n_j n^j = 1 \tag{2.8}$$

The direction, $\mathbf{n} = \mathbf{n}(\tau)$, and so is the *rapidity*, $\phi = \phi(\tau)$. Note that, in spite of the notation, $\phi \in \mathbb{R}$ is not an angle. Comparison with Eq. 2.7 gives the relation between rapidity and the Newtonian velocity

$$\gamma = \cosh\phi, \quad |\mathbf{v}| = c \tanh\phi \tag{2.9}$$

The rapidity is often a convenient representation of the velocity.

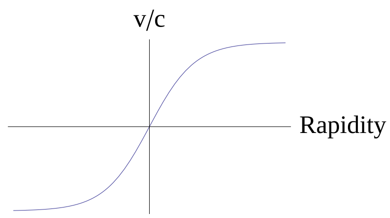


Figure 2.8: The velocity as function of the rapidity

### 2.4.1 4 Acceleration:

The 4-acceleration can be similarly defined as

$$a^\mu = \frac{du^\mu}{d\tau} \tag{2.10}$$

It is always Minkowski orthogonal to the velocity

$$u_\mu a^\mu = 0 \tag{2.11}$$

(Since the Minkowski length of the velocity is constant). It follows that *The 4-acceleration is always a space like vector.*

### 2.4.2 Linear acceleration

In linear acceleration $\mathbf{n}$ is fixed. Differentiating Eq. 2.7 we find for the 4-acceleration

$$a = \dot{u} = c\,\dot{\phi}(\sinh\phi, \mathbf{n}\cosh\phi)$$

Evidently

$$a_\mu a^\mu = c^2 \dot{\phi}^2$$

In particular, constant acceleration, $g$, corresponds to linear dependence of $\phi$ on the proper-time $\tau$

$$\phi(\tau) = \frac{g\tau}{c}$$

In this case we can easily integrate $u = \dot{x}$ to find the world line

$$x(\tau) - x(0) = \frac{c^2}{g}\,(\sinh\phi, \mathbf{n}\,\cosh\phi) \tag{2.12}$$

The world line is a hyperboloid that is eventually tangent to the light cone, as shown Fig. 2.9.

**Remark 2.8.** *By the equivalence principle, you may think of an accelerating observer as an observer in free fall in constant gravitational field. Such an observer has an horizon, like the horizon of a black hole. This is illustrated in the Fig. 2.9.*

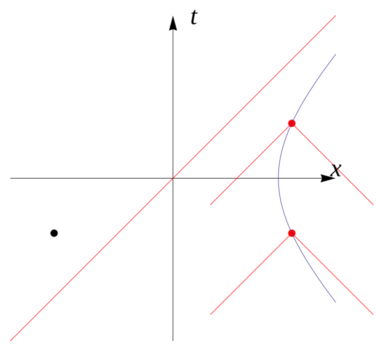


Figure 2.9: An accelerating observer who lives forever, will still see only half the events in Minkowski space-time. He will never see the black dot on the left. The red line is his horizon.

**Remark 2.9** (Numerical coincidence)**.** *An amusing numerical coincidence is that the year, the gravitational acceleration on earth $g$, and $c$ are simply related:* $g \times year/c = 1.03$

**Exercise 2.10.** *What fraction of the velocity of light would you reach in this case.* Answer: $\tanh 1.03 = 0.77$

### 2.4.3 Space travel

You may worry that since $c$ is the ultimate speed, a human being, living for, say 80 years, can explore at most a neighborhood of 80 light-years around earth. This is wrong. The rapidity increases linearly with $T$ and by Eq. 2.12, the distance is exponential in the rapidity. A space traveler who lives for $T$ years, in a space ship which accelerates at $g$ will travel a distances, $\cosh T$, measured in light years. The visible universe has radius of about $10^{11}$ light years. So you will get there is about 26 years.

## 2.5 Cyclotron motion

Consider relativistic circular motion with fixed rapidity $\phi$ and constant acceleration, $a_\mu a^\mu = g^2$. The 4-velocity is

$$u^\mu = c(\cosh\phi, \cos(\omega\tau)\sinh\phi, \sin(\omega\tau)\sinh\phi, 0), \quad \omega c \sinh\phi = g$$

The acceleration is

$$a^\mu = \omega c \sinh\phi(0, -\sin(\omega\tau), \cos(\omega\tau), 0) = g\,(0, -\sin(\omega\tau), \cos(\omega\tau), 0)$$

The orbit is a helix in space-time

$$x(\tau) - x(0) = \frac{c}{\omega}(\omega\tau\cosh\phi, \sin(\omega\tau)\sinh\phi, -\cos(\omega\tau)\sinh\phi, 0)$$

## 2.6 Lorentz transformations

**Definition 1** (Lorentz transformations as isometries of Minkoski space)**.** *The linear transformation $\Lambda$, a $4 \times 4$ matrix, is a Lorentz transformation if it leaves the Minkowski metric $\eta$ invariant*

$$\Lambda^t \eta \Lambda = \eta \Longleftrightarrow {\Lambda_\mu}^\alpha {\Lambda_\nu}^\beta \eta_{\alpha\beta} = \eta_{\mu\nu} \tag{2.13}$$

It follows that

$$\det\Lambda = \pm 1$$

This divides Lorentz transformations into two classes: The proper Lorentz transfromations where $\det\Lambda = 1$. which contain the identity and the improper ones where $\det\Lambda = -1$ that contain the reflection.

**Remark 2.11.** *Lorentz transformations are the analog of orthogonal transformations of Euclidean space.*

### 2.6.1 Space-time translations

Space time translations are trivial Lorentz transformations. They are given by

$$(x')^\mu = x^\mu + a^\mu$$

This gives

$$\Lambda^\mu{}_\nu = \frac{\partial (x')^\mu}{\partial x^\nu} = \delta^\nu_\mu \Longleftrightarrow \Lambda = \mathbb{1}$$

Translations expresses the homogeneity of Minkowski space time.

### 2.6.2 Generators of Lorentz transformations

The Lorentz group is made of rotations and boosts. It is convenient to describe these transformations in terms of their generators. $L$ $L$ is a $4 \times 4$ matrix that generates the one parameter family of transformations

$$\Lambda(t) = e^{Lt}$$

If $\Lambda(t)$ is to be a family of Lorentz transformations it must preserved the Minkowski metric:

$$\eta = \Lambda(t)^t \eta \Lambda(t)$$

Differentiating with respect to $t$ gives the condition that $L$ generates Lorentz transformation

$$\eta L + L^t \eta = 0 \tag{2.14}$$

This relation makes it evident that the generators make a linear space which is spanned by a finite set of linearly independent generators. It turns out that the space has six independent generators: Three rotations and three boosts.

Since $L$ commutes with itself, the associated Lorentz transfromations satisfy a simple addition rule

$$e^{t_1 L} e^{t_2 L} = e^{(t_1+t_2)L} \tag{2.15}$$

This is what you expect for addition of rotations about one fixed axis. But, more surprisingly, this is also the rule for addition of boosts along a fixed axis, *provided we identify $t$ with the rapidity*, as we shall see below.

**Exercise 2.12.** *Show that:*

1. *If $\lambda$ is an eigenvalue of $\Lambda$ so is $\lambda^*$.*
2. *If $\lambda$ is an eigenvalue so is* $1/\lambda$ (Hint: Use $\Lambda^{-1} = \eta \Lambda^t \eta$ to show that $\det(\Lambda - \lambda) = \det(\Lambda^{-1} - \lambda)$ )

### 2.6.3 Rotations

Rotations by $\theta$ about the x-axis and are generated by

$$L_{yz} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{2.16}$$

and a rotation by $\theta$ about the $x$ axis is given by

$$\Lambda_{yz}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{2.17}$$

$L_{yz}$ is is closely related to the angular momentum about the $x$ axis.

**Exercise 2.13.** *Check that $L_{yz}$ satisfies Eq. 2.14*

Similarly for rotations about the $x, y$ space axes. Rotation by $\theta$ about an arbitrary axis **n** is given by

$$e^{\theta(n_x L_{yx} + n_y L_{zx} + n_z L_{xy})} \tag{2.18}$$

The isometry under rotations, expresses the isotropy of space.

**Exercise 2.14.** *Show that*

$$[L_{yz}, L_{zx}] = -L_{xy}, \tag{2.19}$$

Up to factor, $L_{ab} \mapsto -iL_{ab}$, these are the commutation relations of angular momentum

**Remark 2.15.** *An airplane has three rotation controls: Stick, for pitch, rudder for yaw, and ailerons for roll. The three are* linearly *independent, but by the commutation relation you can always generate the third from the first two.*

**Exercise 2.16.** *Calculate the residual rotation of (pairwise-cancelling) rotations*

$$\Lambda_{yz}(\pi/2)\Lambda_{zx}(\pi/2)\Lambda_{yz}(-\pi/2)\Lambda_{zx}(-\pi/2)$$

*Show that this is a rotation about the* $(-1, 1, 1)$ *axis.*

### 2.6.4 Boosts

A boost in the x direction is generated by

$$L_{tx} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.20}$$

The associated Lorentz transformation with rapidity $\phi$ is given by:

$$\Lambda_{tx}(\phi) = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.21}$$

**Exercise 2.17.** *Check that $L_{tx}$ satisfies Eq. 2.14.*

To check that $\phi$ is indeed the rapidity, consider the world-line of the origin. Since

$$\Lambda_{tx}(\phi)(ct, 0, 0, 0)^t = ct(\cosh\phi, \sinh\phi, 0, 0)^t$$

its velocity is $c(\cosh\phi, \sinh\phi, 0, 0)$. Comparison with Eq. 2.7 shows that $\phi$ is the rapidity.

**Exercise 2.18.** *Check Eq. 2.20 in 1-1 dimensional space-time. (Hint: use the formula for $e^{\phi\sigma_x}$ from quantum mechanics.*

One similarly defines the generators of the boost along the $y$ and $z$ axis. A boost by $\phi$ in an arbitrary direction $\mathbf{n}$ is then given by

$$e^{\phi(n_x L_{tx}+n_y L_{ty}+n_z L_{tz})} \tag{2.22}$$

The addition law for velocities in special relativity is a mess. It is, however, simple for the rapidities: Let $\mathbf{n}$ be the direction of motion and $L = (n_x L_{tx} + n_y L_{ty} + n_z L_{tz})$. Since $L$ commutes with itself

$$e^{\phi_1 L} e^{\phi_1 L} = e^{(\phi_1+\phi_2)L}$$

It follows that rapidities (in the same direction) add.

**Exercise 2.19** (Galilean transformations)**.** *Show that for small rapidities Lorentz boosts reduce to Galilean transformation:*

$$t' = t + O(c^{-2}), \quad x' = x - vt + O(c^{-2})$$

### 2.6.5 Commutators

The commutators of the generators of rotations are given by cyclic permutations of Eq. 2.19. The commutators of the generators of rotations with the generators of boosts follow the same rules, since the three generators of boosts are naturally associated with a vector, i.e.

$$[L_{yz}, L_{tx}] = 0, \quad [L_{xy}, L_{tx}] = L_{ty}, \quad [L_{zx}, L_{tx}] = -L_{tz} \tag{2.23}$$

The most interesting and surprising commutator is between two generators of boosts. It turns out that the commutator is a rotation:

$$[L_{tx}, L_{ty}] = L_{xy},$$

This is related to the physical phenomenon known as Thomas precession.

## 2.7 Rotating frames in Minkowski space

You can sometimes hear claims that special relativity can not correctly describe the motion of rotating frames and you need general relativity for this. This is wrong. There is no problem in describing rotating frames, provided they are rotating relatively to Minkowski space-time.

The rotating earth is such a non-inertial frame. Let $\Omega$ be the angular frequency and $(ct', \mathbf{x}')$ be the coordinate of an inertial fame and $(ct, \mathbf{x})$ the coordinates in a rotating frame. A coordinate transformation between the two frames, represented in cylindrical coordinates is

$$t' = t, \quad \rho' = \rho, \quad z' = z, \quad \phi' = \phi + \Omega t$$

The Euclidena metrics are the related by

$$\begin{aligned} d\mathbf{x}'\cdot d\mathbf{x}' &= (d\rho')^2+(dz')^2+\rho'^2(d\phi')^2 \\ &= (d\rho)^2+(dz)^2+\rho^2(d\phi+\Omega dt)^2 \\ &= d\mathbf{x}\cdot d\mathbf{x}+2\rho^2\Omega d\phi dt+\rho^2\Omega^2(dt)^2 \end{aligned}$$

It follows that the Minwoski metric is

$$(-cdt')^2+d\mathbf{x}'\cdot d\mathbf{x}' =$$

$$-(cdt)^2\Big(1-\underbrace{\frac{\Omega^2\rho^2}{c^2}}_{\text{centrifugal}}\Big)+\underbrace{2\Omega\rho^2\,dtd\phi}_{\text{Sangac-Coriolis}}+d\mathbf{x}\cdot d\mathbf{x}$$

**Exercise 2.20.** *$\Omega\rho/c$ at the equator of earth is* $1.5\times10^{-6}$

In the case of earth the centrifugal correction can normally be neglected, but the Sangac term, which is first order in $\Omega$, is important and to a good approximation

$$\begin{aligned} (d\ell)^2 &= -(cdt)^2+2\Omega\rho^2\,dtd\phi+d\mathbf{x}\cdot d\mathbf{x} \\ &= -(cdt)^2+2\Omega\rho^2\,dtd\phi+\rho^2(d\phi)^2+(d\rho)^2+(dz)^2 \end{aligned} \tag{2.24}$$

In particular, if you consider light propagating along the equator $\rho=R,\ z=0$ and, it takes different times for the clockwise and counter clockwise beams to complete a $2\pi$ rotations, given by the two solutions of the quadratic equation for $T_\pm$:

$$0=-(cT_\pm)^2+4\pi\Omega R^2\,T_\pm+R^2(2\pi)^2 \tag{2.25}$$

**Exercise 2.21** (Coordinate times and clock times)**.** *1. Compare the change in* coordinate time $dt$ *in the rotating earth frame with the proper-time $d\tau$ measured by a clock at a fixed location in the rotating frame*

2. *Compare the change in coordinate time $dt'$ in the inertial frame with the change in coordinate time dt in the rotating coordinates*

3. *A clock is taken for a one year trip around earth equator. Show that the time lag relative to a clock that stayed is*

$$\Delta\tau=\pm\frac{2A\Omega}{c^2},\quad A=\pi R_e^2$$

*where $R_e$ is the earth radius and the $\pm$ depends on whether the trip was towards the east or towards the west.*

4. *Compute $\Delta\tau$. (Answer: $207\ [ns]$)*

5. *Explain why the result implies that one can not synchronize clocks on earth.*

6. *Is it still true that $u_\mu u^\mu=-c^2$ in a rotating frame? (*Yes)

7. *What does get modifies is $\gamma$. Show that*

$$\gamma^{-2}=1-(\mathbf{v}/c)^2+2\Omega\dot\phi\rho^2/c^2-\Omega^2\rho^2/c^2$$

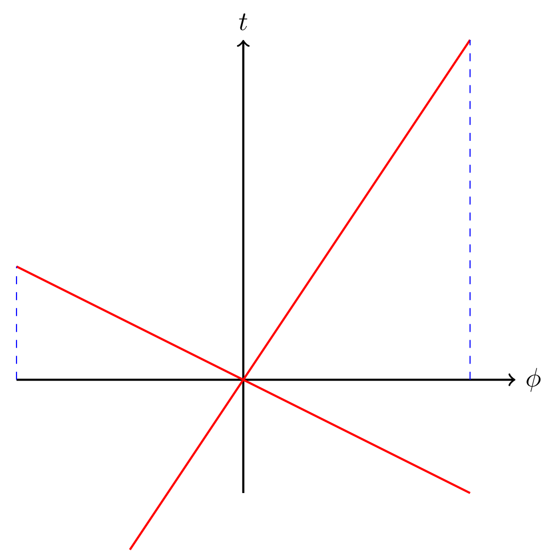


Figure 2.10: Minkowski space in rotating earth coordinates. It takes light different time, in the rotating frame, to complete a clockwise round-trip and counterclockwise round-trip.

### 2.7.1 Rindler coordinates and horizons

Rindler coordinates are the analog of polar coordinate system in a two dimensional Minkowski space time

$$x = \rho \cosh \tau, \quad t = \rho \sinh \tau$$

Since $\cosh \tau \geq \sinh \tau$ the Rindler coordinates cover 1/4 of space time.

**Exercise 2.22.** *Show that*

$$(dx)^2 - (dt)^2 = (d\rho)^2 - \rho^2 (d\tau)^2$$

As we shall see the world line $\rho = const$ describes a uniformly accelerated observer. So the Rindler coordinates are useful in describing accelerated frames in Minkowsky space.

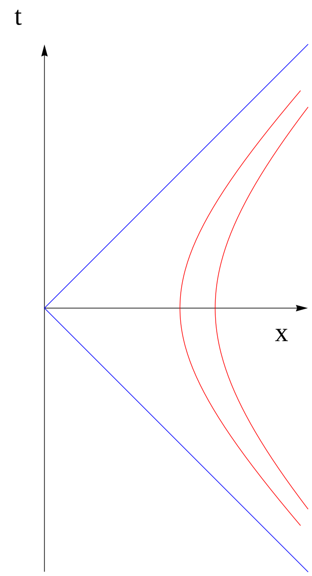


Figure 2.11: The red lines shows the $\rho$ Rindler coordinate.

## 2.8 GPS

Every time you use your GPS and find, with relief, that the GPS really knows where you are, you are testing special and general relativity. It took a century to turn Einstein's revolution into a useful gadget.

GPS works like that: There are about 24 GPS satellites orbiting earth at a radius of about $26,000\,[km]$ and period of about half a day. Their orbit are known (and monitored) with great precision (few centimeters). On each satellite there is an atomic clock that measures its proper time with great precision[1]. Each satellite radios at specific interval, a message that contains its identity and the reading of its clock. Since the orbit of the satellite is known, the data specify the transmission event $X_b^\mu$ of satellite $b$. The corresponding event received by the user is $x^\mu$. Let us focus of the ideal case when both the transmitter and receiver are in empty space so that electromagnetic wave propagate at $c$. It follows that

$$(X_b^\mu - x^\mu)((X_b)_\mu - x_\mu) = 0 \tag{2.26}$$

This is an equation for the 4 unknowns $x_\mu$. To determine the four unknown coordinates $x^\mu$ of the reception event you need 4 equation. You need to receive simultaneously, at least 4 signals from 4 GPS satellites and record 4 transmission events all light-like. One expects that these equations are typically independent and to have a unique solution which is the position and time of thee lost tourist.

[1]To locate a point with a precision of 1 [m] you need to measure time to a precision of $O(10^{-9})[sec]$.

Eq. 2.26 clearly incorporates special relativity as it uses the fact that light propagates at $c$ irrespective of the motion of the satellite and the receiver. It turns out that for GPS to be practically useful, special relativity is not enough. One needs to take account of the slight deviations of space-time from Minkowski. In particular, the self-time of atomic clocks depends, not only on their velocity, but also on the gravitational field that they see and moreover, one needs a better approximation for the metric, that takes into account the gravitational field of eaarth, see section 2.9.) Yet another complication is that in practice one does wants to know the coordinate of the event in the non-inertial coordinate system that rotates with earth (as in section 2.7.) There are many additional complications that need to be taken care of: Atmospheric effects on the velocity of light etc.

Ignoring special or general relativity would degrade the the accuracy of GPS to about 10 km and make it useless. You can then satisfy yourself that special and general relativity has been tested many billions of times.

If you want to know more about GPS, then the article of Neal Ashby in Living Reviews of General Relativity is a good place to learn. Wikipedia is, as usual, quite good as well.

**Exercise 2.23** (Orders of magnitudes)**.**

- *What is a typical velocity of GPS satellite? (Anser: $3.8$ $[km/s]$)*
- *Compute the difference between the coordinate time and the self-time of a GPS clock after one day.* ( $\Delta t \approx \pi R v/c^2 \approx 3.6 \times 10^{-6}$ $[s]$)
- *What is the resulting positioning inaccuracy?*

### 2.8.1 Two dimensional space time

Consider a toy GPS problem in 1+1 dimensions, shown in the Fig. 2.12. If you see 2 satellites in 1+1 dimensions, this means that one is on your right and the other on your left, as in Fig. 2.12. (Otherwise one would eclipse the other.) The light-cones intersect.

**Exercise 2.24** (GPS in 1+1 dimensions)**.** *Two satellites with known orbits ,$\big(a_b^0(\tau), a_b^1(\tau)\big)$, $b = 1, 2$, emit signals at $\tau_a$ and $\tau_b$ respectively. Assuming that $a^\mu - b^\mu$ is space like, show that light-cone intersect at*

$$2x^1 = \pm(a_1^0 - a_2^0) + a_1^1 + a_2^1, \quad 2x^0 = (a_1^0 + a_2^0) \pm (a_1^1 - a_2^1)$$

*(One solution is in the past and the other in the future.)*

This toy model tests:

1. Space-time is approximately Minkowski
2. Electromagnetic waves propagate at $c$
3. The velocity of the satellite at the transmission event is irrelevant
4. The velocity of the lost tourist at the reception event is irrelevant

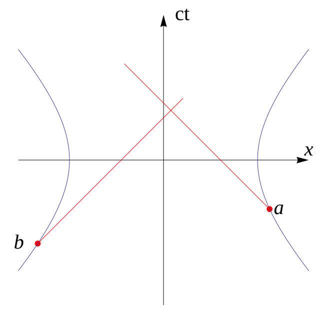


Figure 2.12: The world line of the two satellites are the blue lines. The intersection of the light cones is the event whose coordinates we seek. Since the orbits of the satellites are known, the events $(a^0, a^1)$ and $(b^0, b^1)$ are known given the proper times $\tau_a$ and $\tau_b$.

## 2.9 Space-time near a point mass

Minkowski space-time is a good local approximation of physical space-time. It fails at cosmological distances, at distant times, and when great accuracy is needed. In particular, Minkowski is not a sufficiently approximation of space-time in the theory of GPS (see Neal Ashby in Living Reviews of General Relativity). A more accurate model of the metric in the vicinity of a point mass $M$ is

$$(d\ell)^2 = -(1+\Phi)(cdt)^2 + (1-\Phi)\, d\mathbf{x}\cdot d\mathbf{x}, \quad \Phi(\mathbf{x}) = -\frac{2GM}{c^2|\mathbf{x}|} \tag{2.27}$$

$G$ is Newton constant. $(t, \mathbf{x})$ are space-time coordinates. This metric *is not* equivalent to Minkowski under change of coordinates.

A clock at a fixed location ticks at rate

$$d\tau = \sqrt{1+\Phi}\, dt$$

Far from the star, when $\Phi = 0$, the clock rate coincides with the coordinate time rate. However, close to the star $d\tau < dt$: the clock ticks more slowly than the coordinate time. This means that if you moved near a very massive star, you will outlive your friends who stayed far from it. Kip Thorn reinterpret gravitational attraction as our wish to live longer. On earth, the difference the gravitational metric and Minkoiwski is very small:

**Example 2.25.** *$\Phi$ at the surface of the earth is dominated by the pull of the sun, $\Phi = 2 \times 10^{-8}$ and the pull of the earth is about an order of magnitude smaller due to its pull is $\Phi = -1.4 \times 10^{-9}$. This adds about 1 [sec] to our life expectancy.*

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1. S. Weinberg, Gravitation and Cosmology, Chapters 1 and 2

2. B. F. Schutz, A first course in general relativity, geometric exposition of relativity.

3. Neal Ashby in Living Reviews of General Relativity

# Chapter 3

# The electromagnetic fields

## 3.1 Electromagnetic fields in Minkowski space

The basic objects of mechanics, velocity and accelerations can be viewed as the 3 dimensional shadows of 4-vectors in Minkowski space-time. In both cases the spatial 3-vectors, gained a zero component. 4-vectors then behave much more nicely under Lorentz transformations than their 3-dimensional shadows.

What about the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$? What are they shadows of? From a Euclidean perspective $\mathbf{E}$ and $\mathbf{B}$ are quite different: The force of the electric field is independent of the velocity of the particle and the force of magnetic field is linear in the velocity[1]. In the c.g.s. (=Gaussian) units (which we shall nevertheless use):

$$\mathbf{f} = \underbrace{e\mathbf{E}}_{Coulomb} + \underbrace{e\frac{\mathbf{v}}{c} \times \mathbf{B}}_{Lorentz} \Longrightarrow f_i = eE_i + \frac{e}{c}\varepsilon_{ijk} v^j B^k \tag{3.1}$$

**Remark 3.1** (SI)**.** *In the (defunct) Gaussian (c.g.s.) units the unit of electric field is rather large,* 300 $[V/cm]$*, being three orders of magnitude larger than the field near an A battery, and the unit of magnetic field is Gauss which is rather small, on the scale of the earth magnetic field. The more practical SI units which involve arbitrary constants such as the permittivity* $\epsilon_0 = 8.85 \times 10^{-12} [F/m]$ *and permeability* $\mu_0 = 4\pi \times 10^{-7} [H/m]$*, the unit of electric field is almost five orders of magnitudes smaller,* 1 $[V/m]$*, and the unit of magnetic field is four orders of magnitude larger:* $Tesla = 10^4$ *Gauss, which is comparable to the field near a strong toy magnet. In SI units*[2] *the Coulomb-Lorentz law is*

$$\mathbf{f} = e\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

*The force of electric field of* $1\,[volt/cm]$ *and the magnetic force of* 1 $[Gauss]$ *have comparable magnitudes at velocities of* $1000\,[km/sec]$*.*

[1] Moreover, since $\mathbf{f}, \mathbf{v}$ are vectors so is $\mathbf{E}$ and under inversion $\mathbf{E} \mapsto -\mathbf{E}$. But $\mathbf{B}$ is a pseudo-vector: Under inversion $\mathbf{B} \mapsto \mathbf{B}$.

[2] And also if one takes units where $c = 1$.

The partition into electric and magnetic field is, of course, different in different inertial frames[3].

**Exercise 3.2** (Galilei invariance)**.** *In Newtonian mechanics the force is Galilean invariant:* $\mathbf{f} = \mathbf{f}'$ (Why). *Show that the transformation rules for the fields under Galilean transformations with relative velocity* $\mathbf{v}$ *is*

$$\mathbf{E}' = \mathbf{E} + \frac{1}{c}\,\mathbf{v} \times \mathbf{B}\,, \quad \mathbf{B}' = \mathbf{B} \tag{3.2}$$

The mixing of $\mathbf{E}$ and $\mathbf{B}$ under the change of inertial frames suggests that they come from a single entity in space-time. This entity *is not* a 4-vector since we need 6 slots and a 4-vector has too few. It is not a general second rank tensor, since it has 16 components which are too many. It is not even a symmetric second rank tensor since it has 10 components, still too many. An anti-symmetric rank 2 tensor

$$F_{\mu\nu} = -F_{\nu\mu} \tag{3.3}$$

has 6 components, which is just right.

**Exercise 3.3** (Symmetry is Lorentz invariant)**.** *Symmetry is a tensorial invariant, e.g. if* $F_{\mu\nu}$ *is anti-symmetric so is* $(F')_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}$ *under arbitrary change of coordinates (and Lorentz transformation in particular). As a consequence if* $F$ *is anti-symmetric in Cartesian coordinates it is also anti-symmetric in spherical coordinates.*

This leaves us with the problem of how to put the two vector fields $(\mathbf{E}, \mathbf{B})$ in $F_{\mu\nu}$. The quickest way to do this is to write the Coulomb-Lorentz law in a way that is manifestly Lorentz invariant, i.e. as an identity between 4-vectors, which reduces to its non-relativistic form in the approximation $\gamma \approx 1$. Since the Coulomb-Lorentz force in linear in the fields and in the velocity, we can get a 4-force vector by contracting the field tensor with the 4-velocity

$$f_\mu = \frac{e}{c} F_{\mu\nu} u^\nu \Longrightarrow f_1 = \frac{e}{c}\left(F_{10}u^0 + \cancel{F_{11}u^1} + F_{12}u^2 + F_{13}u^3\right) \tag{3.4}$$

Since $u = \gamma(c, \mathbf{v})$ in the non-relativistic approximation $\gamma \approx 1$ we identify the electric field with the first row of $F$

$$F_{10} = E_1 \Longrightarrow F_{j0} = E_j \tag{3.5}$$

and the magnetic field with

$$F_{12} = B^3 \Longrightarrow F_{jk} = \epsilon_{jki} B^i \tag{3.6}$$

In conclusion the identification of $F$ with $E$ and $B$ is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B^z & -B^y \\ E_y & -B^z & 0 & B^x \\ E_z & B^y & -B^x & 0 \end{pmatrix} \tag{3.7}$$

[3] Einstein was lead to the discovery of special relativity y considering the symmetry of Maxwell's equations.

**Exercise 3.4** (Mixed components). *The matrix associated with the mixed tensor F is neither symmetric nor anti-symmetric. Verify that*

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{3.8}$$

**Remark 3.5** (Coordinate free form). *In a coordinate free form*

$$\mathbf{F} = F_{\mu\nu}\, \mathbf{e}^{\mu} \otimes \mathbf{e}^{\nu}$$

Eq. 3.4 gives the relativistic Coulomb-Lorentz force 4-vector.

### 3.1.1 Anti-symmetric tensors describe a pair of vectors

More generally, given any anti-symmetric tensor

$$F = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & y^3 & -y^2 \\ -x_2 & -y^3 & 0 & y^1 \\ -x_3 & y^2 & -y_1 & 0 \end{pmatrix} \tag{3.9}$$

the triplets $(x_1, x_{2,3}\,)$ and $(y^1, y^2, y^3)$ transform like vectors under 3 dimensional rotations considered as a subgroup of the Lorentz group.

#### Euclidean Rotations

Consider a 3×3 rotation matrix $R$ of Euclidean space so $R^{-1} = R^t$. In Euclidean space contravariant components are the same as covariant components. The components of matrix $R$ are

$$R \underbrace{\;j\;}_{row} \overbrace{\;k\;}^{column} = (R^t) \overbrace{\;k\;}^{row} \underbrace{\;j\;}_{column}$$

The lift of $R$ to a rotation in Minkowski space is

$$\Lambda_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}, \tag{3.10}$$

The (covariant) components $x_j$ transfrom by

$$\begin{aligned} (x')_j = F'_{0j} &= \Lambda_0{}^{\mu} \Lambda_j{}^{\nu} F_{\mu\nu} \\ &= R_j{}^k F_{0k} \\ &= R_j{}^k x_k \end{aligned} \tag{3.11}$$

which is the rule of transformation of Euclidean 3-vectors under rotations. Now consider the triplet $2y^j = \varepsilon^{jmn}F_{mn}$. The Levi-Civita symbol is invariant under (proper) rotations since $\det R = 1$. Hence

$$\begin{aligned}(2y')^j &= (\varepsilon')^{jmn}(F')_{mn}\\ &= \varepsilon^{jmn}R_m{}^a R_n{}^b F_{ab}\\ &= \varepsilon^{jmn}\varepsilon_{abk}R_m{}^a R_n{}^b\, y^k\\ &= (R^{-1})_k{}^j\,(2y)^k \end{aligned}\tag{3.12}$$

The last step uses the formula for inverse of a $3\times 3$ matrix $R$

$$(R^{-1})_k{}^j = \frac{1}{2\det R}\,\varepsilon_{kab}\varepsilon^{jmn}R_m{}^a R_n{}^b$$

and the fact that $\det R = 1$ for a rotation. It remains to get rid of the inverse. To do that write

$$(R^{-1})_k{}^j = (R^t)_k{}^j = R^j{}_k = R_j{}^k$$

(in Cartesian coordinates). This gives

$$\begin{aligned}(2y')^j &= (R^t)_k{}^j\,(2y)^k\\ &= R^j{}_k\,(2y)^k\end{aligned}\tag{3.13}$$

One sees that $\mathbf{x}$ and $\mathbf{y}$ both transform as vectors. More precisely, $\mathbf{x}$ is a vector while $\mathbf{y}$ is a pseudo-vector as the transformation rule relied on the use of Levi-Civita (and $\det R = 1$).

**Example 3.6.** *The covariant components of the field tensor in cylindrical coordinates $(ct, \rho, \phi, z)$ are*

$$F_{cylind} = \begin{pmatrix} 0 & -E_x c - E_y s & \rho(-E_y c + E_x s) & -E_z\\ \dots & 0 & \rho B_z & -B_y c + B_x s\\ \dots & \dots & 0 & \rho(B_x c + B_y s)\\ \dots & \dots & \dots & 0\end{pmatrix}$$

*where $c = \cos\phi$ and $s = \sin\phi$. In normalized coordinates*

$$\begin{pmatrix} 0 & -E_x c - E_y s & -E_y c + E_x s & -E_z\\ \dots & 0 & B_z & -B_y c + B_x s\\ \dots & \dots & 0 & B_x c + B_y s\\ \dots & \dots & \dots & 0\end{pmatrix}$$

**Exercise 3.7** (Magnetic field of currnet line)**.** *A line of current $I$ along the z-axis creates a magnetic field $\mathbf{B} = \frac{2I}{c\rho}\hat\theta$ in cylindrical coordinates. Show that in cylindrical coordinates $F_{\rho z} = -F_{z\rho} = \frac{2I}{c\rho}$ and all other components vanish. (*Hint: It is simpler to use properties of the basis vectors $\mathbf{e}^\rho, \mathbf{e}^z$ rather than compute the transformation matrix.)

## 3.2 The field of a uniformly moving charge

For a charge $e$ at rest at the origin

$$E_j = e\frac{x_j}{r^3}, \quad B_j = 0 \tag{3.14}$$

Everything is time independent so we can compute this at any time we want.

Consider the Lorentz boost

$$\Lambda = \begin{pmatrix} C & S & 0 & 0 \\ S & C & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C = \cosh\phi,\ S = \sinh\phi \tag{3.15}$$

where $\phi$ is the rapidity connected with the usual $\gamma$ and $\beta$ by

$$\gamma = \cosh\phi, \quad \beta = \tanh\phi \tag{3.16}$$

The two Lorentz scalars are

$$E^2 - B^2 = \frac{e^2}{r^4}, \quad E\cdot B = 0 \tag{3.17}$$

The transformation rules are

$$F'_{\mu\nu}(x') = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}(x') = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta F_{\alpha\beta}(\Lambda x) \tag{3.18}$$

In the frame where we see a moving charge, everything depends on time. So let us compute everything at $t' = 0$ when the charge is at the origin. We have

$$x = Cx', \quad y = y', \quad z = z' \tag{3.19}$$

In particular

$$\begin{aligned} r^2 &= C^2x'^2 + y'^2 + z'^2 = C^2x'^2 + (C^2 - S^2)(y'^2 + z'^2) \\ &= C^2r'^2 - S^2(y'^2 + z'^2) \end{aligned} \tag{3.20}$$

Let us turn to the fields. $E_x$ does not change

$$E'_x = F'_{01} = \Lambda_0{}^\alpha \Lambda_1{}^\beta F_{\alpha\beta} = (\Lambda_0{}^0\Lambda_1{}^1 - \Lambda_0{}^1\Lambda_1{}^0)F_{01} = (C^2 - S^2)E_x = E_x \tag{3.21}$$

Hence, at $t' = 0$

$$E_x(x) = e\frac{x}{r^3} = e\gamma\frac{x'}{r^3(r')} = E'_x(x') \tag{3.22}$$

where $r(r')$ is the ugly expression Eq. (3.20).

For the transverse directions

$$E'_y = F'_{02} = \Lambda_0{}^\alpha \Lambda_2{}^\beta F_{\alpha\beta} = \Lambda_0{}^\alpha F_{\alpha 2} = CF_{02} = \gamma E_y \tag{3.23}$$

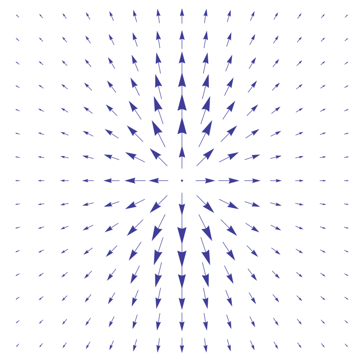

Figure 3.1: The vector field of a moving charge with rapidity $\phi = 1$. The field is manifestly radial but not spherically symmetric.

and so

$$E_y(x) = e\frac{y}{r^3}, \quad = E'_y(x') = \gamma e\frac{y}{r^3} = \gamma e\frac{y'}{r^3(r')} \tag{3.24}$$

The formula is the same but for different reasons. In one case $\gamma$ came form the field transformation and in the other from the coordinates.

It now follows that in both frames the field is radial, because

$$\frac{x}{y} = \frac{E_x(x)}{E_y(x)} = \frac{E'_x(x)}{E'_y(x)} = \frac{x'}{y'} \tag{3.25}$$

**Remark 3.8.** *This is a bit surprising. One could have argued that since the field is radial in the rest frame, you may expect it to point in the direction of the particle at the retarded time, not now.*

The total strength of the field is

$$E'^2 = e^2\frac{\gamma^2}{r^4} = \gamma^2 E^2 \tag{3.26}$$

It is stronger in the frame where the charge is seen moving (computed for the same event).

## 3.3 Lorentz scalars

One can construct interesting Lorentz scalars from the field tensor $F$. There are no interesting scalars that are linear in the field at a given point since

$$F_{\mu\nu}\eta^{\mu\nu} = F_\mu{}^\mu = 0$$

We can however construct interesting quadratic scalars:

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= F_{0j}F^{0j} + F_{j0}F^{j0} + F_{jk}F^{jk} \\ &= -2E_jE_j + 2B_iB_i \\ &= -2(\mathbf{E}^2 - \mathbf{B}^2) \end{aligned} \tag{3.27}$$

You might be worried that we have made a sign error: $\mathbf{E}^2+\mathbf{B^2}$ is proportional to the energy density of the field. Why the minus sign? Actually, it is a fortunate that we did not get the energy density, for, as we shall see the energy density is a component of a second rank tensor.

The minus sign is reminiscent of the minus sign you find in Lagrangian mechanics: The Lagrangian is the difference of the kinetic and potential energies, not their sum. As we shall see, this is not a coincidence.

### 3.3.1 Duality and Levi-Civita

Duality, denoted by $*$, is an operation whose square in the identity: $** = \mathbb{1}$. Taking a dual involves no loss of information. The Levi-Civita tensor allows us to define such a duality between anti-symmetric tensors[4]. In 2 dimensions, the duality is between anti-symmetric tensors and scalars:

$$\omega^* = \frac{1}{2}\varepsilon^{jk}\omega_{jk} \Longleftrightarrow (\omega^*)_{jk} = \varepsilon^{jk}\omega \tag{3.28}$$

and in 3 dimensions between vectors and anti-symmetric 2-rank tensors:

$$(\omega^*)^j = \frac{1}{2}\varepsilon^{ijk}\omega_{jk} \tag{3.29}$$

In 4-dimensions it is between anti-symmetric tensors and corresponds to exchanging the two 3-vectors associated to the tensor:

$$(F^*)^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{3.30}$$

**Remark 3.9.** *In any dimension there is a duality between anti-symmetric tensors of rank $r$ and anti-symmetric tensors of rank $n-r$. Anti-symmetric tensors of rank $r$ make a linear space whose dimension $\binom{n}{r}$ and since $\binom{n}{r} = \binom{n-r}{r}$ the two linear spaces are isomorphic. The contraction of the Levi-Civita with an anti-symmetric tensor of rank $r$ gives an anti-symmetric tensor of rank $n-r$.*

**Exercise 3.10.** *Show that*

*1.*

$$\varepsilon_{\alpha\beta\gamma\delta}\varepsilon^{\alpha\beta\mu\nu} = 2(\delta_\alpha{}^\mu\delta_\beta{}^\nu - \delta_\alpha{}^\nu\delta_\beta{}^\mu)$$

[4]For the sake of simplicity, we shall take $|\det g| = 1$ so that Levi-Civita tensor and symbol coincide. In general, duality depends on the metric as it involves $\det g$.

2. *Using this show that*

$$F^{**} = F$$

Duality effectively interchanges $\mathbf{E}$ and $-\mathbf{B}$.

$$(F^*)^{0i} = \tfrac{1}{2}\varepsilon^{0ijk}F_{jk} = \varepsilon^{ijk}F_{jk} = B^i, \quad \{ijk\} \in \text{even permutation of } \{1,2,3\}.$$

Similarly,

$$(F^*)^{jk} = \tfrac{1}{2}\varepsilon^{jk\alpha\beta}F_{\alpha\beta} = \varepsilon^{jk0i}F_{0i} = \varepsilon^{0jki}F_{0i} = -\varepsilon^{jki}E_i$$

**Exercise 3.11.** *Write the formulas in this section without assuming* $|\det g| = 1$.

### 3.3.2 Second Lorentz scalar

We can construct a second Lorentz scalar by contracting $F$ with its dual $F^*$

$$(F^*)^{\mu\nu}F_{\mu\nu} = 2(F^*)^{0j}F_{0j} + (F^*)^{jk}F_{jk} = -4\mathbf{E}\cdot\mathbf{B} \tag{3.31}$$

Electromagnetic fields up to scaling, give nine equivalence classes:

$$\mathbf{E}^2 - \mathbf{B}^2 = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \qquad \mathbf{E}\cdot\mathbf{B} = \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

For example, if there is electric field and no magnetic field in one frame, then in any other frame $\mathbf{E}^2 - \mathbf{B}^2 > 0$ and $\mathbf{E}\cdot\mathbf{B} = 0$ etc. Similarly, the field of a plane electromagnetic wave has $\mathbf{E}^2 - \mathbf{B}^2 = \mathbf{E}\cdot\mathbf{B} = 0$, in any frame.

**Exercise 3.12.** *Prove of give a counter example to:* $\mathbf{E}^2 - \mathbf{B}^2 > 0$ *and* $\mathbf{E}\cdot\mathbf{B} = 0$ *imply that there is a Lorentz frame where* $\mathbf{B} = 0$

**Exercise 3.13.** *How does the formula* $\mathbf{E}\cdot\mathbf{B}$ *change when* $|\det g| \neq 1$.

## 3.4 The homogeneous Maxwell equations

The two homogeneous Maxwell's equations are

$$\nabla\cdot\mathbf{B} = 0, \quad \partial_0\mathbf{B} + \nabla\times\mathbf{E} = 0, \quad \partial_0 = \frac{1}{c}\partial_t \tag{3.32}$$

The first says that there are no magnetic monopoles and the second is the Faraday law of induction. Together, they give 4 equations.

The equations have a coordinate independent form but assume inertial frame. This means that the metric tensor in space-time has the form

$$g = \begin{pmatrix} -1 & 0 \\ 0 & g_3 \end{pmatrix} \tag{3.33}$$

where $g_3$ is a time-independent $3 \times 3$ positive matrix.

Since I do not wish to enter into writing equations with covariant derivatives, we shall proceed assuming Cartesian coordinates, i.e. assuming $g_3 = \mathbb{1}$.

The two equations 3.32 have different character. Farady's law is an evolution equation that dictates how $\mathbf{B}$ evolves in time, while the no-monopole condition is not an evolution equation. It can be viewed as a constraint equation on the admissible magnetic fields at any given time. The two equations are not independent, and must be consistent in the sense that if $\mathbf{B}$ starts divergence-less, it must evolve in a way that its stays divergence-less. This is indeed the case:

$$\partial_0\big(\nabla \cdot \mathbf{B}\big) = -\nabla \cdot \big(\nabla \times \mathbf{E}\big) = 0 \tag{3.34}$$

The no-monopole condition may therefore be viewed as a constraint on the initial data.

### 3.4.1 No monopoles

The absence of magnetic monopoles is expressed by

$$\nabla \cdot \mathbf{B} = \partial_j(B^j) = 0 \tag{3.35}$$

This can be written in terms of $F$ as

$$0 = \partial_j(B^j) = \partial_j(F^*)^{0j} = \partial_\mu(F^*)^{0\mu} = \partial_\mu(F^*)^{\mu 0} \tag{3.36}$$

**Exercise 3.14.** *Generalize this to* $|\det g| \neq 1$

### 3.4.2 Faraday law

Faraday law of induction written in components

$$0 = \partial_0 B^i + \epsilon^{ijk}\partial_j E_k = \partial_0(B^i) + \epsilon^{ijk}\partial_j E_k = \partial_\mu(F^*)^{\mu i} \tag{3.37}$$

### 3.4.3 Amalgamating the homogeneous Maxwell equations

The 4 homogeneous Maxwell equations can therefore be amalgamated into a single equation for a vector field

$$\partial_\mu(F^*)^{\mu\nu} = 0 \tag{3.38}$$

This can be phrased as a conservation law for $F^*$, or as the statement that $F^*$ is divergence free.

## 3.5 Potentials

The homogeneous Maxwell's equations can be rephrased as the statement that the fields are derived from potentials. $\mathbf{B}$ is derived from the vector potential $\mathbf{A}$:

$$\nabla \cdot \mathbf{B} = 0 \Longleftrightarrow \mathbf{B} = \nabla \times \mathbf{A} \tag{3.39}$$

Similarly, $\partial_0 A + \mathbf{E}$ being derived from a scalar potential $\phi$ is a consequence of combining Faraday with the no monopole condition:

$$0 = \partial_0 \mathbf{B} + \nabla \times \mathbf{E} = \nabla \times (\partial_0 \mathbf{A} + \mathbf{E}) \Longleftrightarrow \partial_0 A + \mathbf{E} = -\nabla\phi \tag{3.40}$$

You normally see this equation rearranged so the field is on one side and the potentials on the other side.

### 3.5.1 The 4-potential

We want to amalgamate $\phi, \mathbf{A})$ into a single 4-vector $A$. Since $F$ is an anti-symmetric tensor which is related to derivatives of the potential a good starting point is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.41}$$

Comparing with Eq. 3.39 we see that $\mathbf{A} = (A_1, A_2, A_3)$ and comparing with Eq. 3.40 we find $A_0 = -\phi$. In summary

$$A_\mu = (-\phi, \mathbf{A}) \tag{3.42}$$

We can now reformulate the 4 homogeneous Maxwell's equations concisely as

$$\partial_\alpha (F^*)^{\alpha\beta} = 0 \tag{3.43}$$

Indeed,this follows from

$$2\partial_\alpha (F^*)^{\alpha\beta} = \varepsilon^{\alpha\beta\mu\nu}\partial_\alpha F_{\mu\nu} = \varepsilon^{\alpha\beta\mu\nu}(\partial_{\alpha\mu} A_\nu - \partial_{\alpha\nu} A_\mu) \tag{3.44}$$

and observing that a contraction of a symmetric and anti-symmetric objects vanish

$$\varepsilon^{\alpha\beta\mu\nu}\partial_{\alpha\mu} = 0 \tag{3.45}$$

Eq. 3.43 follows.

**Exercise 3.15.** *[Bianchi identity] Show that 4 equations (3.43) can also be written as*

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$$

In mechanics the price for using the a non-inertial coordinate system, such as earth, leads to the price of the emergence of new forces: Coriolis and Centrifugal. You may wonder if there is an analog in electrodynamics. The next exercise explains why there is none.

**Exercise 3.16.** *Consider the coordinate system attached to the rotating earth introduced in the previous chapter.*

1. *Compute* $\det g$ *for the earth rotating coordinate system.* (Answer: $\det g = -1$)

2. *What does this imply about the homogeneous Maxwell equations in the earth frame?*

**Exercise 3.17** (Harmonic $A$)**.** *Given the vector potential*

$$A_\mu(x) = A_\mu e^{ik_\nu x^\nu}$$

*Show that*

$$F_{\mu\nu}F^{\mu\nu} = -2\left(k_\mu k^\mu\, A_\nu A^\nu - (k_\mu A^\mu)^2\right), \quad F_{\mu\nu}(F^*)^{\mu\nu} = 0$$

*It follows that if $k^\mu$ is light like the first term in the brackets on the left vanish. The second term in the brackets vanishes if $k$ and $A$ are Lorentz orthogonal.*

## 3.6 Gauge transformations

The vector potential

$$A_\mu = \partial_\mu \Lambda \tag{3.46}$$

for any (differentiable) function $\Lambda(x^0, \dots, x^\mu$ is called a pure-gauge. The associated field $F$ vanishes identically:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_{\mu\nu}\Lambda - \partial_{\nu\mu}\Lambda = 0 \tag{3.47}$$

Since the (linear) mapping $A \mapsto F$ has a large kernel–the pure gauge fields, it has no inverse and $F$ does not determine the 4-potential $A$ uniquely. For any (scalar) function $\Lambda$ of space-time

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{3.48}$$

give the same field $F$. Fields have a direct physical meaning: they can be measured as forces. Potentials are tools for computations and no physical instrument measure potentials. In particular, a voltmeter does not measure $\phi$.

### 3.6.1 Non-local gauge invariants Lorentz scalars

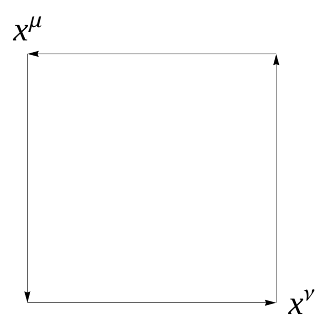


Figure 3.2: Loop and surface element for Stokes.

Although $A$ is not gauge invariant, line integral of $A$ over closed loops in space-time is a gauge invariant scalar. By Stokes:

$$\oint A_\mu dx^\mu = \int F_{\mu\nu} dS^{\mu\nu}$$

where $dS$ is the area element spanned by the loop. Both sides are manifestly Lorentz scalars, and the right hand side is manifestly gauge invariant.

A familiar, special case of the formula is a closed loop at fixed time, $dx^\mu = (0, dx^k)$, where

$$\oint A_k dx^k = \oint \mathbf{A} \cdot d\ell = \int \nabla \times \mathbf{A} \cdot dS = \int \mathbf{B} \cdot dS = \Phi$$

gives the magnetic flux through the loop.

**Exercise 3.18.** *Suppose that $\gamma$ is a curve in Euclidean space and consider the surface $S$ spanned by the curve for $t \in [0, t_0]$. Show that $\int F dS$ is the emf action, i.e. $\int \mathcal{E}_{emf}\ dt$.*

Quantum mechanics gives a fundamental unit of magnetic flux $\Phi_0 = 2 \times 10^{-15}$ $[Weber]$. In SI $[Weber] = [\hbar/e]$. There are about 10 quantum flux quanta flux through a 1 $[\mu^2]$ area of the earth magnetic field. So for a bacterium a quantum flux is a natural magnetic flux scale. It is interesting, and even mysterious, that when quantum mechanics meets special relativity, the Lorentz scalars give rise to quantized objects. Here are some examples whey $\Phi_0$ shows up: It is the quantized magnetic flux in vortices that thread a superconductor; The charge of magnetic monopoles; and it is the emf action $\int \mathbf{E} \cdot dt$ that shows up in the quantum Hall effect.

**Exercise 3.19.** *What is $\Phi_0 = 2 \times 10^{-15}$ $[Weber]$ in c.s.g in terms of $\bar{,}e$ and $c$ .*

## 3.7 Electromagnetic fields in curvilinear coordinates

In the case that we want to describe the electromagnetic fields in non-Cartesian, curvilinear coordinates we need to pay attention to the distinction between covariant and contarvariante components of **E** and **B** and between the Levi-Civita tensor and symbol. In particular we need to adjust Eq. 3.1

$$f_i = eE_i + \frac{e}{c}\sqrt{|g|}\varepsilon_{ijk} v^j B^k \tag{3.49}$$

where $g$ is as in Eq. 3.33. It follows that

$$F_{j,0} = E_j, \quad F_{jk} = \sqrt{|g|}\varepsilon_{ijk} B^i \tag{3.50}$$

We retain Eq. 3.41

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.51}$$

and define the dual so that $F^*$ is a tensor, (not a density). This means that we replace Eq. 3.30 by

$$(F^*)^{\mu\nu} = \frac{1}{2\sqrt{|g|}}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{3.52}$$

From the rhs of Eq.3.44 we have the Bianchi-like identity

$$0 = \partial_\mu(\sqrt{|g|}(F^*)^{\mu\nu}) \tag{3.53}$$

The 0-component reads

$$\begin{aligned}
0 &= \partial_\mu(\sqrt{|g|}(F^*)^{0\mu}) \\
&= \partial_i(\sqrt{|g|}(F^*)^{0i}) = \frac{1}{2}\partial_i(\varepsilon^{0ijk}F_{jk}) \\
&= \frac{1}{2}\partial_i(\varepsilon^{ijk}F_{jk}) \\
&= \partial_i(\sqrt{|g|})B^i) \\
&= \nabla\cdot\mathbf{B}
\end{aligned} \tag{3.54}$$

which is, as before, the no-monopole condition in curvilinear coordinates.

Similarly, for the i-th spatial component

$$\begin{aligned}
0 &= \partial_\mu(\sqrt{|g|}(F^*)^{i\mu}) \\
&= \partial_0(\sqrt{|g|}(F^*)^{i0}) + \partial_j(\sqrt{|g|}(F^*)^{ij}) + \partial_k(\sqrt{|g|}(F^*)^{ik}) \\
&= \frac{1}{2}\partial_0(\varepsilon^{i0jk}F_{jk}) + \partial_i(\varepsilon^{ij0k}F_{0k}) + \partial_k(\varepsilon^{ik0j}F_{0j}) \\
&= -\frac{1}{2}\partial_0(\varepsilon^{ijk}F_{jk}) + \partial_i(\varepsilon^{ijk}F_{0k}) - \partial_k(\varepsilon^{ijk}F_{0j}) \\
&= \partial_0(\sqrt{|g|}B^i) + \varepsilon^{ijk}(\partial_i E_k - \partial_k E_j) \\
&= \partial_0(\sqrt{|g|})B^i) + \sqrt{|g|}(\nabla\times\mathbf{E})^i \\
&= \sqrt{|g|}\big(\partial_0\mathbf{B} + \nabla\times\mathbf{E}\big)^i
\end{aligned} \tag{3.55}$$

This gives Faraday law. In the last line we used the fact that curvilinear coordinates for Maxwell equations in an inertial frame, have a $g$ which is time-independent.

It is quite remarkable that Electrodynamics in curvilinear coordinates can be formulated with no reference to Christoffel symbols.

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# Chapter 4

# Variational principle

The equations of motion of a relativistic (charged) particle are formulated as a variational principle.

## 4.1 Physics is where the action is

Lagrangian mechanics reformulates Newtonian mechanics: Newton equations are interpreted as the Euler-Lagrange equations—The minimizers of the action. The advantages of the Lagrangian formulation are

- It spares the trouble of finding the components of possibly complicated vectors by giving the problem a scalar formulation involving the action
- It is manifestly independent of the choice of coordinates, and allows for using even non-inertial coordinate systems (and automatically takes care of fictitious, d'Alambert, forces.)
- It is convenient for handling symmetries and the associated conservation laws.
- It can be generalized to the theory of Electromagnetism, to Gravitation and Quantum Mechanics. Essentially, all known theories admit a Lagrangian formulation
- It is elegant

The property of being minimizer does not depend of the the choice of coordinates and so the Lagrangian formulation guarantees the tensorial character of the equations. Since Lorentz transformations are special coordinate transformations of Minkowski space, a theory is guaranteed to be Lorentz invariant once the action is a Lorentz scalar.

In this section, we shall see how one formulates relativistic mechanics using Lagrangian formalism. The algorithm for doing that is:

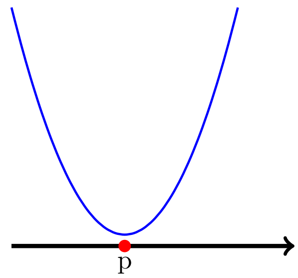



Figure 4.1: The minimum at the point $p$ does not depend on what coordinates you choose for the x axis)

- Find a Lorentz scalar that could serve as an action
- Verify that in the non-relativistic limit the action reduces to its Newtonian form

This does not fix the relativistic action uniquely. This is a feature that allows for creativity to be involved.

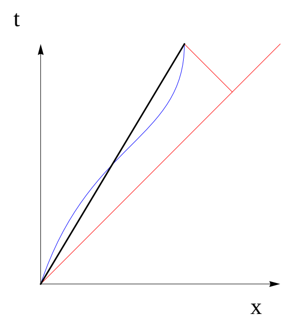



Figure 4.2: The world line is required to have time-like tangents–for the action to be real. The blue curve represents the variation of a world line with fixed end points. The black line maximizes the proper-time $\tau$. The red lines are light-like. The red path between the end points has zero proper-time.

### 4.1.1 Action for a free massive particle

Massive particles move at subluminal speed. A natural Lorentz scalar associated to a path in space-time of such a particle is the elapse of proper time $\tau$. We could have used the proper time as a candidate for the relativistic action. However, we also want the relativistic action to reduce to the non-relativistic action for slow particle. So, we need to massage the proper time a bit.

We start by fixing the dimensions. Traditionally, the action has units of $[p][x]$, we append scalars $c$ and $m$ to the proper time fix the dimension:

$$S_p = -mc^2 \int \underbrace{d\tau}_{proper\ time} = \int L \underbrace{dt}_{coordinate\ time} \tag{4.1}$$

The Lagrangian

$$\begin{aligned} L &= -\frac{mc^2}{\gamma} \\ &= -mc^2(1 - \mathbf{v}^2/c^2)^{-1/2} \\ &\approx -\underbrace{mc^2}_{const} + \underbrace{\frac{m}{2}\mathbf{v}\cdot\mathbf{v}}_{kinetic\ energy} \end{aligned} \tag{4.2}$$

This is the same as the classical Lagrangian of free particle, up to a (large) negative constant $-mc^2$. Adding a constant to the action does not affect the minimizing path, of course.

**Exercise 4.1.** *In rotating (earth) coordinates, 2.7,*

$$\begin{aligned} \frac{1}{\gamma} &\approx 1 - \frac{1}{2c^2}\left(\mathbf{v}^2 - \Omega^2\rho^2 + \Omega\,\dot{\phi}\rho^2\right) \\ &= 1 - \frac{1}{2c^2}\left(\underbrace{\mathbf{v}^2}_{Kinetic} - \underbrace{(\mathbf{\Omega}\times\mathbf{x})^2}_{centrifugal} + \underbrace{2\,\mathbf{\Omega}\cdot\mathbf{v}\times\mathbf{x}}_{Coriolis}\right) \end{aligned}$$

*Use this to drive the equations of motion of a free particle in a rotating frame.*

### 4.1.2 Interaction with the electromagnetic field

Consider now a particle that interacts with an external electromagnetic field. In classical mechanics we describe the Coulomb and Lorentz forces by adding to the action two terms

$$S_{int} = -e\int \phi(x)dt + \frac{e}{c}\int \mathbf{A}(x)\cdot\mathbf{v}\,dt \tag{4.3}$$

where $\phi$ is the scalar potential and $\mathbf{A}$ the vector potential (and I have chosen c.g.s. units). We can write the action as a Lorentz scalar by introducing the Minkowski gauge field

$$A_\mu = (-c\phi(x), \mathbf{A}(x)) \tag{4.4}$$

The interaction is the Lorentz scalar

$$S_{int} = \frac{e}{c}\int A_\mu dx^\mu \tag{4.5}$$

### 4.1.3 Gauge invariance

Since the action is constructed from the potentials, it is not manifestly gauge invariant. One needs to worry about gauge invariance. Under change of gauge

$$A'_\mu = A_\mu - \partial_\mu \Lambda \tag{4.6}$$

This leads to

$$S'_{int} = S_{int} - \frac{e}{c}\int (\partial_\mu \Lambda)\, dx^\mu = S_{int} - \frac{e}{c}\big(\Lambda(x_f) - \Lambda(x_i)\big) \tag{4.7}$$

Although $S_{int}$ changes under a gauge transformation, the change is fully determined by the end points of the orbit. Variations of the orbit that keep the endpoints fixed do not affect the the action. This guarantees that the Euler Lagrange equations are gauge invariant.

### 4.1.4 Euler-Lagrange

$S$ is a Lorentz scalar, of course. If we want to use the ready made Euler-Lagrange equations, familar from mechanics,

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{x}} \tag{4.8}$$

we nee to write

$$S = \int L dt, \quad L = L(\mathbf{x}, \mathbf{v}) \tag{4.9}$$

The relativistic Lagrangian $L$ is given by

$$\gamma L = -mc^2 + \frac{e}{c} A_\mu u^\mu \tag{4.10}$$

When Euler-Lagrange equations are appllied to $L$ one find the correct relativistic equations of motions. But the formulas above suffer from the deficiency that formulation is not covariant: $\gamma L$ is a scalar, but $L$ is not. This reflects the choice of lab-time in Eq. 4.9. For this reason we shall re-derive the variation equation in a covariant fashon in the next section.

## 4.2 Variation of the action

The action is a function on paths: It associates a number with a given path $x^\mu(\tau)$. We may think of the path as parametrized by its proper time. The action clearly the form

$$S = \int_{x_i}^{x_f} f(x, u) d\tau \tag{4.11}$$

In fact

$$f(x, u) = -mc^2 + \frac{e}{c} A_\mu(x) u^\mu$$

The end point (events) $x_i$ and $x_f$ are fixed.

We shall denote the variation of the path by

$$\delta x = \{ \underbrace{\delta x^0(\tau), \ldots \delta x^3(\tau)}_{infinitesimal\ functions} \}$$

$\delta x$ vanishes at the end points. The strategy one uses to derive the Euler-Lagrange equations is to use integration by parts to bring $\delta S$ to the form

$$\delta S = h_\mu(x,u)\delta x^\mu\Big|_{x_i}^{x_f} + \int_{x_i}^{x_f} g_\mu(x,u,\dot{u})\delta x^\mu d\tau$$

Since $\delta x^\mu$ vanish at the end points the boundary term drops. And since $\delta x^\mu$ are arbitrary,[1] the action is extremal if

$$g_\mu(x,u,\dot{u}) = 0$$

These are Euler-Lagrange equations and the derivation is manifestly covariant. This discussion masks the subtlety that a variation of the path entails a variation of the proper-time, which is the integration variable.

### 4.2.1 Variation of the action of a free particle

When the path is varied, the proper time $d\tau$ is affected. The variation in the proper-time with is given by

$$\delta(cd\tau)^2 = 2c^2 d\tau\delta(d\tau) = \delta(-dx_\mu dx^\mu) = -2dx^\mu\,\delta(dx_\mu) \tag{4.12}$$

Factoring $2d\tau$ we can write this in terms of the 4-velocity as

$$c^2\delta(d\tau) = -u^\mu\,\delta(dx_\mu) = -d\big(u^\mu\,\delta x_\mu\big) + du^\mu\,(\delta x_\mu)$$

The variation of $S_p$ for a free particle is then

$$-\delta S_p = -m\,u_\mu\delta x^\mu\big|_{end\ pts} + m\int \dot{u}_\mu\,\delta x^\mu\,d\tau \tag{4.13}$$

Euler-Lagrange equation follow by setting the boundary term and the variation to zero. This gives

$$m\dot{u}_\mu = 0 \tag{4.14}$$

Recall that the momentum in classical mechanics is defined as the rate of change of the action due to change of the end-point of a classical path. This means that we look at the boundary term when we set $m\dot{u} = 0$ in the integral in Eq.4.13, namely

$$\delta S_p = p_\mu\delta x^\mu = m\,u_\mu\delta x^\mu \tag{4.15}$$

The covariant components of the momentum are the gradient of the action with respect to the end points.

[1] Viewed as functions of $\tau$ the variations satisfy the constraint: $(d\delta x_\mu)(d\delta x^\mu) = -(cd\tau)^2$

### 4.2.2 Variation of the action associated to interaction

For $S_{int}$ write

$$\delta(A_\mu dx^\mu) = (\delta A)_\mu dx^\mu + A_\mu \delta(dx)^\mu = (\partial_\nu A_\mu)\,(\delta x^\nu) dx^\mu + A_\mu \delta(dx)^\mu$$

The basic idea in the calculus of variation is always to use integration by parts to get rid of terms of the form $\delta dx$. Hence, rewrite the last term

$$A_\mu \delta(dx)^\mu = d\big(A_\mu \delta x^\mu\big) - (dA)_\mu \delta x^\mu = d\big(A_\mu \delta x^\mu\big) - (\partial_\nu A_\mu) dx^\nu\, \delta x^\mu$$

Combining the two expressions and changing summation indices where needed we find

$$\begin{aligned}\delta(A_\mu dx^\mu) &= d\big(A_\mu \delta x^\mu\big) + (\partial_\mu A_\nu)\,(\delta x^\mu) dx^\nu - (\partial_\nu A_\mu) dx^\nu\, \delta x^\mu \\ &= d\big(A_\mu \delta x^\mu\big) + \Big(\partial_\mu A_\nu - \partial_\nu A_\mu\Big) u^\nu\, d\tau\, \delta x^\mu \\ &= d\big(A_\mu \delta x^\mu\big) + F_{\mu\nu}\, u^\nu\, d\tau\, \delta x^\mu\end{aligned}$$

Hence,

$$\delta S_{int} = \frac{e}{c}\big(A_\mu \delta x^\mu\big)|_{bdry} + \frac{e}{c}\int F_{\mu\nu}\, u^\nu\, d\tau\, \delta x^\nu \tag{4.16}$$

where we introduced the second rank tensor $F_{\mu\nu}$ to describe the electromagnetic fields fields $\mathbf{E}$ and $\mathbf{B}$.

### 4.2.3 Euler-Lagrange equation

The variation of the total action vanishes for any $\delta x^\mu$ provided the integrand in $S_{free} + S_{int}$ add to zero. This gives the Euler-Lagrange equation

$$\dot p_\mu = m\, \dot u_\mu = \frac{e}{c} F_{\mu\nu} u^\nu \tag{4.17}$$

we have guessed in the previous section.

**Exercise 4.2** (Charged particle in a constant fields)**.** *Solve the equations of motion of a charge particle in constant parallel electric and magnetic fields*

**Exercise 4.3** (Charged particle in a radiation field)**.** *Show that the equations of motion of a charge particle in the radiation field of a circularly polarized plane wave admit solutions that are circular orbit in the plane orthogonal to the direction of propagation of the light.*

### 4.2.4 The non-relativistic limit

It is also easy to check that Eq. 4.17 reduces to the standard formulas of non-relativistic classical mechanics. For a slow particle has

$$u^\mu \approx (c, \mathbf{v}), \quad \dot u_j = a_j$$

(recall that $\eta = (-1, 1, 1, 1)$). The Euler-Lagrange equations reduce to Newton equations of motions

$$m\mathbf{a} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{4.18}$$

provided we identify

$$\mathbf{E}_j = F_{j0}, \quad \mathbf{B}_i = \frac{1}{2}\varepsilon^{ijk} F_{jk} \tag{4.19}$$

### 4.2.5 The minimiser of the action

In general the action may not have an reasonable minimizer and when it does, the minimizer need not be unique. The (non-relativistic) Harmonic oscillator is an example. The Lagrangian is

$$2L = \dot{x}^2 - x^2 \tag{4.20}$$

Consider paths that starts and terminates at the origin in half the period, $t = \pi$. The Euler-Lagrange equation are satisfied by

$$x(t) = A \sin t$$

for arbitrary amplitude $A$. All solve the equation of motion $\ddot{x} = -x$ and so are local minimizers. For all of these the action vanishes:

$$S = \int_0^\pi L dt = A^2 \int_0^\pi (\cos^2 t - \sin^2 t) dt = 0,$$

The action has a positive contribution from the kinetic energy and a negative contribution from the potential energy. Each of the two can be arbitrarily large. This suggests that $S$ is actually unbounded below when considered as a function of paths. This is indeed the case.

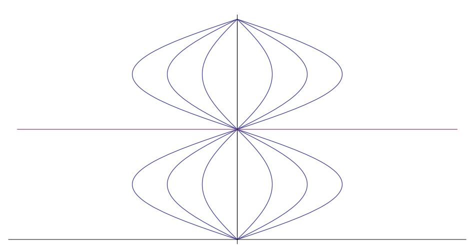


Figure 4.3: There are infinitely many paths connecting the origin when the time difference is half the period. But there is no honest minimizer connecting the origin to any other point on the red line at half the period. The minimizer "goes through infinity".

To see this consider paths that connects $x = 0$ at time $t = 0$ with $x_0$ at time $t = \pi$. A family of such paths is

$$x(t) = A \sin \omega t, \quad \omega = 1 - \frac{1}{\pi} \arcsin(x_0/A)$$

The action is

$$S = \int_0^\pi L dt = A^2 \int_0^\pi (\omega^2 \cos^2 \omega t - \sin^2 \omega t) dt \xrightarrow[A\to\infty]{-} 2x_0 A$$

We see that indeed $S$ is unbounded below.

### Convexity and uniqueness

A function $S$ is called convex if

$$S(\lambda x + \lambda' y) \le \lambda S(x) + \lambda' S(y), \quad \lambda + \lambda' = 1, \quad 1 \ge \lambda, \lambda' \ge 0$$

For example, the function in Fig. 4.1 is convex. It is evident that if a function is convex its minimum is unique. (It may, however, lie at infinity).

The notion of convexity extends to the case that $x$, the argument of $S$, is itself a function–a path. $\gamma$ is a convex function of $\mathbf{v}$. It follows that the action is a convex function of the path. This then implies that the minimizer for a free relativistic particle is unique.

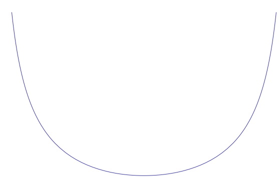


Figure 4.4: $\gamma$ is a convex function of $\mathbf{v}$ . This implies that $S$ is a convex functional of the path.

## 4.3 Geodesics in Curved space-time. (You may want to skip this)

In a general space-time (such as in section 2.9) the notion of proper-time is defined by

$$(cd\tau)^2 = -g_{\mu\nu}(x)\, dx^\mu dx^\nu \tag{4.21}$$

The 4-velocity $\dot{u}^\mu$ is still normalized to $-c^2$ since

$$(cd\tau)^2 = -g_{\mu\nu}(x)\, u^\mu u^\nu\, (d\tau)^2 = -u_\mu u^\mu\, (d\tau)^2$$

**Remark 4.4.** *The acceleration is orthogonal to the velocity in Minkowski space, but not in the more general case when the metric is position dependent. In fact,*

$$0 = \frac{d}{d\tau}(u^\mu g_{\mu\nu} u^\nu) = 2\dot{u}^\mu u_\mu + (\partial_\alpha g_{\mu\nu}) u^\mu u^\nu u^\alpha = 0 \tag{4.22}$$

We want to find the path that *minimizes* the action (equivalently, maximizes the proper time)

$$S = -mc^2 \int d\tau$$

The variation of $(cd\tau)^2$ is

$$\delta(cd\tau)^2 = 2c^2(d\tau)\delta(d\tau) = -(\delta g_{\mu\nu})\, dx^\mu dx^\nu - 2g_{\mu\nu}\, \delta(dx^\mu)dx^\nu$$

Hence

$$-2c^2\delta(d\tau) = (\delta g_{\mu\nu})\, u^\nu\, dx^\mu + 2g_{\mu\nu}\, u^\nu\, \delta(dx^\mu) \tag{4.23}$$

Rewrite the first term on the right as

$$(\delta g_{\mu\nu})\, u^\nu dx^\mu = (\partial_\alpha g_{\mu\nu})\, u^\nu dx^\mu\, \delta x^\alpha = (\partial_\mu g_{\alpha\nu})\, u^\nu dx^\alpha\, \delta x^\mu$$

The second term can be rewritten as

$$\begin{aligned} g_{\mu\nu}\, \delta(dx^\mu)u^\nu &= d(g_{\mu\nu}\, \delta x^\mu u^\nu) - d(g_{\mu\nu}\, u^\nu)\, \delta x^\mu \\ &= d(g_{\mu\nu}\, \delta x^\mu u^\nu) - (\partial_\alpha g_{\mu\nu})\, dx^\alpha\, u^\nu\, \delta x^\mu - g_{\mu\nu}\, du^\nu\, \delta x^\mu \end{aligned}$$

collecting and factoring 2

$$-c^2\delta(d\tau) = d(g_{\mu\nu}\, \delta x^\mu u^\nu) + \Big(\tfrac{1}{2}(\partial_\mu g_{\alpha\nu})\, u^\nu dx^\alpha - (\partial_\alpha g_{\mu\nu})\, u^\nu\, dx^\alpha - g_{\mu\nu}\, du^\nu\Big)\delta x^\mu$$

The first term is a boundary term which does not contribute to the variation of the action. Dividing by $d\tau$ the brackets on the right, and renaming the dummy index $\nu, \beta$ give

$$g_{\mu\nu}\, \dot{u}^\nu + \Big(\partial_\alpha g_{\mu\beta} - \tfrac{1}{2}\partial_\mu g_{\alpha\beta}\Big)\, u^\beta\, u^\alpha = 0 \tag{4.24}$$

The equation has the form

$$\dot{u}^\mu + \Gamma^\mu_{\alpha\beta}\, u^\alpha\, u^\beta = 0 \tag{4.25}$$

where $\Gamma$ is linear in the derivatives of $g$. Since only the symmetric part of $\Gamma^\mu_{\alpha\beta}$ contributes, we define $\Gamma^\mu_{\alpha\beta}$ so it is explicitly symmetric

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}g^{\mu\nu}\Big(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta}\Big) \tag{4.26}$$

This is the celebrated Christoffel symbol.

**Exercise 4.5.** *Show that great circles on the sphere are geodesics.*

**Exercise 4.6** (Geodesic equation in covariant components)**.** *Show that the geodesic equation for the covariant components satisfies the equation*

$$2\dot{u}_\mu = -\Big(\partial_\mu g^{\alpha\beta}\Big)\, u_\alpha\, u_\beta$$

### 4.3.1 Relativistic Kepler law

For a non-relativistic planet orbiting a sun in a circular orbit, equating the centrifugal force with the gravitational attraction gives

$$\omega^2 R = \frac{GM}{R^2} \tag{4.27}$$

This is Kepler third law which relates the periods of all planets in the solar system with their radii

$$T^2 \propto R^3 \tag{4.28}$$

Let us now see how one can get a Kepler type relation for geodesics of a diagonal, time-independent, metric in $2+1$ dimensions with circular symmetry, i.e. $g = g(\rho)$:

$$-(cd\tau)^2 = g_t(cdt)^2 + g_\rho(d\rho)^2 + \rho^2(d\theta)^2 \tag{4.29}$$

For a circular, stationary geodesic

$$u^\rho = \dot\rho = 0, \quad u^\theta = \dot\theta = \omega, \quad u^t = c\dot t = c\gamma \tag{4.30}$$

with $\omega$ and $\gamma$ constant in time. For a time-like $u$ we have

$$-c^2 = u_\mu u^\mu = (c\gamma)^2 g_t + \omega^2\rho^2 \tag{4.31}$$

Now, since $\dot u^\mu = 0$ for a stationary geodesic, the geodesic (differential) equations reduce to constraints relating $\rho, \omega, \gamma$. As $u^\mu$ has only two non-zero components the 4 geodesic equations are

$$0 = c^2\Gamma^\mu_{tt}\gamma^2 + 2c\Gamma^\mu_{\theta t}\omega\gamma + \Gamma^\mu_{\theta\theta}\omega^2 \tag{4.32}$$

As $g$ is a function of $\rho$ the formula for the Christoffel symbols simplify[2]

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2}g^\mu\Big(\cancel{\partial_\alpha g_{\mu\beta}} + \cancel{\partial_\beta g_{\mu\alpha}} - \partial_\mu g_{\alpha\beta}\Big), \quad \alpha,\beta \in t,\theta \tag{4.33}$$

The (non-zero) Christoffel symbols are $\Gamma^\rho_{tt}$ and $\Gamma^\rho_{\theta\theta}$ so the geodesic equation gives a single constraint relating $\omega, \gamma$ and $\rho$

$$0 = \Gamma^\rho_{tt}(c\gamma)^2 + \Gamma^\rho_{\theta\theta}\omega^2 \tag{4.34}$$

Combined with Eq. 4.31 which relates $\gamma$ and $\omega$, we get a relation between $\rho$ and $\omega$ for circular (time-like) stationary orbits

$$0 = -\Gamma^\rho_{tt}(\rho)(c^2 + \omega^2\rho^2) + g_{tt}\Gamma^\rho_{\theta\theta}(\rho)\omega^2 \tag{4.35}$$

Solving for $\omega^2$ we get Kepler type law (in proper time)

$$\omega^2_\tau = \frac{c^2}{g_{tt}\Gamma^\rho_{\theta\theta}/\Gamma^\rho_{tt} - \rho^2} \tag{4.36}$$

[2] When $g$ is diagonal, we denote its diagonal elements by $g_\mu$

It is more physical to measure $\omega$ in coordinate time, i.e

$$\omega_t^2 = \left(\frac{d\tau}{dt}\right)^2 \omega_\tau^2 = \frac{-c^2 g_t + \rho^2 \omega_t^2}{g_{tt}\Gamma^\rho_{\theta\theta}/\Gamma^\rho_{tt} - \rho^2} \tag{4.37}$$

where we used the metric and orbit to relate self and coordinate time.

$$-(cd\tau)^2 = g_t(cdt)^2 + \rho^2(\omega_\tau d\tau)^2 = g_t(cdt)^2 + \rho^2(\omega_t dt)^2 \tag{4.38}$$

Solving for $\omega_t$ we get:

$$\omega_t^2 = c^2 \frac{\Gamma^\rho_{tt}}{\Gamma^\rho_{\theta\theta}} \tag{4.39}$$

The ratio of the Christoffel symbols can be computed from

$$\Gamma^\rho_{tt} = -\frac{1}{2} g^\rho \partial_\rho g_{tt}, \quad \Gamma^\rho_{\theta\theta} = -\frac{1}{2} g^\rho \partial_\rho g_{\theta\theta} = -\frac{1}{2} g^\rho \partial_\rho \left(\rho^2\right) = -\rho g^\rho \tag{4.40}$$

Hence

$$\frac{\Gamma^\rho_{\theta\theta}}{\Gamma^\rho_{tt}} = \frac{2\rho}{\partial_\rho g_t}$$

This gives for Kepler's law

$$\omega_t^2 = c^2 \frac{\partial_\rho g_t}{2\rho} \tag{4.41}$$

independent of $g_\rho$. In the case of the Schwartzschild metric

$$g_t = -(1 + \Phi), \quad \Phi = -\frac{2GM}{c^2 \rho} \tag{4.42}$$

Kepler's law takes the familiar form

$$\omega_\tau^2 = \frac{GM}{\rho^3} \tag{4.43}$$

Interestingly, the appears to be no GR correction to Kepler's law. GR is hidden in the relation between the self-time and coordinate time. Substituting Kepler in Eq. 4.38 we get

$$(d\tau)^2 = \left(1 - \frac{3GM}{\rho}\right)(dt)^2 \tag{4.44}$$

so the computation only makes sense for

$$\frac{3GM}{\rho} < 1$$

## 4.4 Supplement

### 4.4.1 Fermat principle

The mother of variational principles is Fermat principle. It formulates geometric optics at the the minimizer of the time of propagation between two points. The ray propagates in a medium with index of refraction $n(\mathbf{x})$. The propagation time $dt$ is $c\,dt = n(\mathbf{x})|d\mathbf{x}|$. We can think of $n$ as inducing a metric in Euclidean space–one that measures the propagation time:

$$(c\,dt)^2 = n^2(\mathbf{x})\,d\mathbf{x}\cdot d\mathbf{x} = n^2(\mathbf{x})\,(d\ell)^2$$

Such a metric is called conformal.

Consider a ray $\mathbf{x}(\ell)$ parametrized by it Euclidean length: $d\ell = \sqrt{d\mathbf{x}\cdot d\mathbf{x}}$. The tangent to the ray

$$\mathbf{t} = \frac{d\mathbf{x}}{d\ell}$$

is a unit vector. For a variation of the path $\delta\mathbf{x}$:

$$\delta(d\ell) = \frac{d\mathbf{x}\cdot d\delta\mathbf{x}}{d\ell} = \mathbf{t}\cdot d\delta\mathbf{x}, \quad \delta n = (\delta\mathbf{x}\cdot\nabla)n$$

The corresponding time variation is (the integral of)

$$\delta(c\ dt) = (\delta\mathbf{x}\cdot\nabla)n d\ell + n\mathbf{t}\cdot\ d\delta\mathbf{x} = \delta\mathbf{x}\cdot\Big(\nabla n\,d\ell - d\big(n\mathbf{t}\big)\Big) + d\big(n\mathbf{t}\cdot\ \delta\mathbf{x}\big)$$

The integral of variation vanishes provided the brackets (and boundary terms) vanish:

$$\frac{d(n\mathbf{t})}{d\ell} = \nabla n$$

In particular, in a region where the refraction index is a constant, $\nabla n = 0$, the ray keeps it direction of propagation: $\mathbf{t}$ is a constant.

**Exercise 4.7.** *Show that the equation of motion is consistent with the* $\mathbf{t}$ *being a unite vector.*

**Exercise 4.8.** *Derive Snell's law, fig.4.5,*

$$n_1\sin\theta_1 = n_2\sin\theta_2$$

*from Fermat principle.*

### 4.4.2 Rainbow

The simplest features of the rainbow can be understood from Snell's law.

**Exercise 4.9.** *Use Snell law and show that a light ray in air ($n_a = 1$) hitting a water droplet, $n_w > 1$, at lattitude $\theta$ is reflected back at angle $2\alpha(\theta) = 4\phi(\theta) - 2\theta$, see figure. The function $\phi(\theta)$ is defined by Snell law: $n_w\sin\phi = \sin\theta$.*

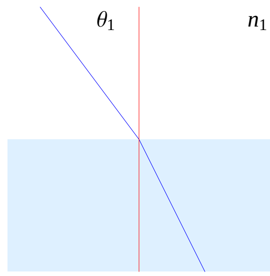



Figure 4.5: The change in direction of a ray when $n$ jumps is ddetermined by Snell's law

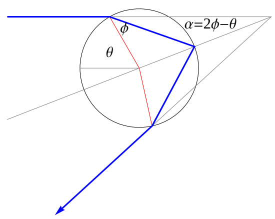



Figure 4.6: The blue line shows a ray undergoing one internal reflection in a drop of water. The impact angle $\theta$ is defined in the figure. The outgoing ray is focused near the maximum $2\phi - \theta$. This partial focusing is called a caustic. This gives the direction of the rainbow.

The intensity of the light reflected at angle $I(2\alpha)$ is proportional to the intensity of the incoming light:

$$d(\sin\theta) = I(2\alpha)\,|d\alpha|$$

A computation gives

$$\frac{d\alpha}{d\theta} = -1 + 2\frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}$$

The derivative vanishes for

$$3\cos^2\theta = n^2 - 1$$

which gives a real value for $\theta$ provided $1 < n < 2$. This gives the maximal value of $2\alpha$. Evidently, $I(2\alpha) = \infty$ there. The divergence implies focusing of the reflected light. This is called *caustic* in geometrical optics.

**Exercise 4.10.** *Show that for water* $(n = 1.33)$ *the caustic occurs for* $2\alpha = 42°$. *This is the main angle of the rainbow, first found by Bacon in 1268. (Different colors have slightly different angles due to the slight frequency dependence of* $n$*).*

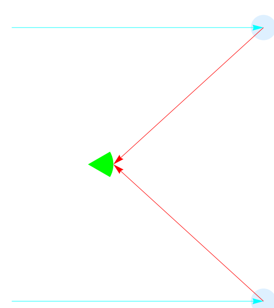


Figure 4.7: The cyan arrows represent light rays from the sun. The two small light-blue balls represent two water droplets. The red arrow are the reflected light rays in the direction of the rainbow caustics. The green eye represents the observer. Pilots sometimes see rainbow that are circular.

# Chapter 5

# Currents

The sources of electromagnetic fields are the charges and currents. To describe the sources in a Lorentz invariant framework we need to amalgamate the non-relativistic notions of charge and currents into a single notion in space-time the 4-current.

## 5.1 Charge densities and currents

For a point charges $e$ moving with the trajectories $\boldsymbol{\xi}(t)$ in 3 dimensions, the non-relativistic notion of charge density $\rho$ and current $\mathbf{j}$ is defined by

$$\rho(\mathbf{x},t) = e\delta^{(3)}\big(\mathbf{x}-\boldsymbol{\xi}(t)\big), \quad \mathbf{j}(\mathbf{x},t) = e\delta^{(3)}\big(\mathbf{x}-\boldsymbol{\xi}(t)\big)\mathbf{v}(t) \tag{5.1}$$

where $\mathbf{v} = \dot{\boldsymbol{\xi}}$ is the velocity. For several particles, of charges $e_a$, moving with trajectories $\xi_a$, the generalization is

$$\rho(\mathbf{x},t) = \sum_a e_a\delta^{(3)}\big(\mathbf{x}-\boldsymbol{\xi}_a(t)\big), \quad \mathbf{j}(\mathbf{x},t) = \sum_a e_a\delta^{(3)}\big(\mathbf{x}-\boldsymbol{\xi}_a(t)\big)\mathbf{v}_a(t) \tag{5.2}$$

We would like to amalgamate $\rho$ and $\mathbf{j}$ into a notion of a 4-current-density. However, neither $\mathbf{v}$ nor $\delta^{(3)}(\mathbf{x})$ are natural objects in space-time and, a-priori, you may well worry that the non-relativistic notions of charge and currents are, at best, a non-relativistic approximations so we will need to tinker with them before being able to amalgamate them. It will turn out that these expressions are fine as they stand. (We still need to adjust dimensions so we can fit both $\rho$ and $\mathbf{j}$ in a 4-vector with identical dimensions.)

It is clear that is is enough to formulate the notions of current and density for a single particle. This simplifies the notation.

### 5.1.1 4-current-density

Consider a point particle whose trajectory is given $\xi^\mu(\tau)$ as a function of its proper-time. For the sake of simplicity, we work in *Minkowski Cartesian* coordinates. We can make a 4-current density using only scalars 4-vectors and in general, objects that behave nicely under Lorentz transformations:

$$j^\mu(x) = ec \int d\tau \ \delta^{(4)}\big(x - \xi(\tau)\big) \ \dot{\xi}^\mu(\tau) \tag{5.3}$$

where dot is a derivative with respect to the proper time. The scalar factor $ec$ fixes the dimensions to the dimensions of current density.

**Remark 5.1.** *$d\tau$ is a scalar, and $\dot{\xi}^\mu$ a 4-vector. The delta function is a density, i.e. under a coordinate transformation it is multiplied by a power of $\det g$. Indeed, under scaling $x' = \lambda x$, the metric transforms as $\lambda^2 g'_{\mu\nu} = g_{\mu\nu}$, so in $d$ dimensions $\lambda^{2d} \det g' = \det g$. Since $\delta^d(x') = \delta^d(\lambda x) = \lambda^{-d}\delta(x)$ we see that $\delta$ is a density as it transforms by $\frac{1}{\sqrt{|g'|}}\delta(x') = \frac{1}{\sqrt{|g|}}\delta(x)$. The current density is a density as its name suggest.*

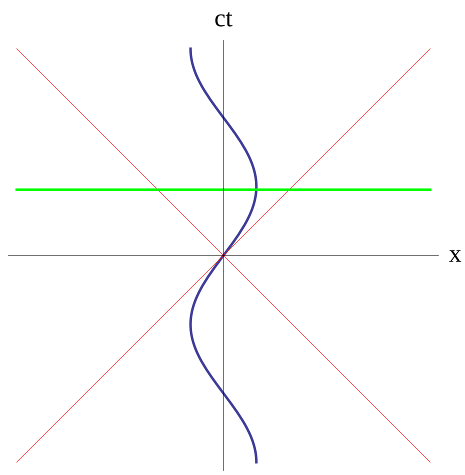



Figure 5.1: The parametrized orbit $\xi^\mu(\tau)$.

To relate this expression to Eq. (5.2) integrate over $\tau$. This gets rid of one of the delta functions. Since $\xi$ is a real orbit, there is a 1-1 correspondence between coordinate time $\xi^0(\tau)/c$ and the proper time $\tau$. Changing variables from $cd\tau$ to $d\xi^0$

$$\begin{aligned} c\int d\tau \ \delta^{(4)}\big(x - \xi(\tau)\big)\dot{\xi}^\mu(\tau) &= c\int d\xi^0 \ \delta^{(4)}\big(x - \xi\big)\dot{\xi}^\mu \frac{d\tau}{d\xi^0} \\ &= \int d\xi^0 \ \delta^{(3)}\big(\mathbf{x} - \boldsymbol{\xi}(\xi^0)\big)\delta(ct - \xi^0)v^\mu(\xi^0) \\ &= \delta^{(3)}\big(\mathbf{x} - \boldsymbol{\xi}(t)\big)v^\mu(t) \end{aligned}$$

where $v^\mu = (c, \mathbf{v}) = \gamma u^\mu$. In conclusion, we get for the 4-vector of current density

$$j^\mu = (c\rho, \mathbf{j})$$

The result is a pleasant surprise because it essentially coincides what we knew from non-relativstic physics. $\delta^{(3)}$ is not a density in Minkowsky space, and $v^\mu$ is not a 4-vector. However, together they conspire to give the 4-vector (density) $j^\mu$. There are no relativistic corrections one needs to make to the classical formulas for charge densities and currents.

### 5.1.2 Charge conservation

Charge conservation says that the total charge, at any given instant

$$\int d^3x\rho(\mathbf{x},t) = e\int d^3x\delta^{(3)}(\mathbf{x}-\boldsymbol{\xi}(t)) = e$$

is independent of the time slice $t$.

**Remark 5.2.** *It is also independent of the Lorentz frame in which the slice is taken.*

Charge conservation is expressed by the continuity equation. Consider the rigid transport of a bump function: $\rho(\mathbf{x}-\boldsymbol{\xi}(t))$. Evidently, the total charge is conserved

$$\int d^3x\rho(\mathbf{x}-\boldsymbol{\xi}(t)) = \int d^3x\rho(\mathbf{x}) \tag{5.4}$$

To derive te continuity equation write $\dot{\boldsymbol{\xi}} = \mathbf{v}$. Then

$$\partial_t\rho\big(\mathbf{x}-\boldsymbol{\xi}(t)\big) = -(\mathbf{v}\cdot\nabla_x)\rho\big(\mathbf{x}-\boldsymbol{\xi}(t)\big) \tag{5.5}$$

Since $\mathbf{v}(t)$ is not a function of $\mathbf{x}$

$$\nabla_x\cdot\big(\mathbf{v}(t)\rho(\mathbf{x}-\boldsymbol{\xi}(t))\big) = (\mathbf{v}\cdot\nabla_x)\rho\big(\mathbf{x}-\xi(t)\big)$$

This, together with Eq. 5.5 gives the continuity equation

$$0 = \partial_t\underbrace{\rho}_{density} + \nabla\cdot\underbrace{(\mathbf{v}\rho)}_{current} = \partial_\mu j^\mu, \quad j^\mu = (c,\mathbf{v})\rho \tag{5.6}$$

A point charge is the limit $\rho\to\delta$.

Once this equation holds for one charge, it holds for any number. When we consider huge numbers of charges with poor spatial resolution we may then think of $j^\mu(x)$ as a smooth function on space time, which satisfies the continuity equation.

**Example 5.3** (Orders of magnitudes)**.**

1. *A current of 1 Ampere transports* $6\times10^{18}$ *electrons per second*
2. *A copper wire of cross section* $S = 1\,[mm^2]$ *has* $8\times10^{20}\,[cm^{-1}]$ *atoms per unit length. Since copper has valence 2, the number of electrons per unit length is* $\approx 1.6\times10^{21}\,[cm^{-1}]$*. The classical formula for the current* $I = env$

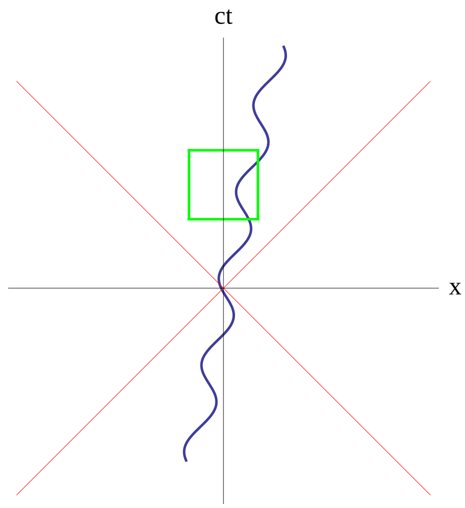



Figure 5.2: Charge conservation expresses the fact that the orbit is a continuous curve which does not terminate and moves always into the future. If it enters a box in space-time it also leaves it. If the orbit enters the box at the bottom leaves it at the top we say that the charge in the box is conserved. If it leaves and enters on the sides we say that incoming current balances the outgoing current.

*gives very small velocities, about 40 $[\mu/sec]$. This classical computation is misleading. In reality, only a small fraction of the electrons, near the Fermi energy, participate in the conductance, and these electrons actually move at the Fermi velocity, which is quite large (about $c/137$). To treat the problem honestly we need quantum mechanics.*

### 5.1.3 Current conservation and gauge invariance

We want to generalize the expression for $S_{int}$ form a finite collection of charges to a continuous distribution. For a single charge the action representing the interaction is

$$\begin{aligned} S_{int} &= \frac{e}{c}\int A_\mu(\xi)u^\mu(\tau)\,d\tau \\ &= \frac{e}{c^2}\int A_\mu(\xi)u^\mu(\tau)\,\delta^{(3)}(\mathbf{x}-\boldsymbol{\xi}(\tau))\;d^3x\,d(c\tau) \\ &= \frac{e}{c^2}\int A_\mu(\xi^0,\mathbf{x})v^\mu(\xi^0)\,\delta^{(3)}(\mathbf{x}-\boldsymbol{\xi}(t))\;d^3x\,d\xi^0 \\ &= \frac{e}{c^2}\int A_\mu(x)v^\mu(t)\,\delta^{(3)}(\mathbf{x}-\boldsymbol{\xi}(t))\;d\Omega, \quad d\Omega = d^4x \end{aligned}$$

The middle two terms are the 4-current, hence

$$S_{int} = \frac{1}{c^2} \int A_\mu(x) j^\mu(x)\, d\Omega \tag{5.7}$$

describing both smooth and discrete 4-current distributions.

### 5.1.4 Gauge invariance and the continuity equation

We have seen that under a change of gauge $S_{int}$ for a single charge changed by a boundary term. This then implied the gauge invariance of the Euler-Lagrange equations. It is interesting to reconsider this issue for smooth current distributions. We will learn something. Under a change of gauge

$$A_\mu \to A_\mu + \partial_\mu \Lambda$$

The interaction action changes by

$$S_{int} \to S_{int} - \frac{1}{c^2} \int (\partial_\mu \Lambda)\, j^\mu d\Omega \tag{5.8}$$

The integrand can be rearranged as

$$(\partial_\mu \Lambda)\, j^\mu = \partial_\mu(\Lambda\, j^\mu) - \Lambda\, \partial_\mu\, (j^\mu))$$

The first term gives a boundary term and vanishes if $\Lambda j^\mu \to 0$ at infinity. $S_{int}$ is therefore guaranteed to be gauge invariance provided the current satisfies the continuity equation

$$\partial_\mu j^\mu = 0 \tag{5.9}$$

The expresses the intimate relation between gauge invariance and charge conservation.

### 5.1.5 The continuity equation in curvilinear coordinates

It is instructive to generalize the derivation of the continuity equation to curvilinear coordinates. The main change is that now

$$d\Omega = \sqrt{|g|} d^4 x \tag{5.10}$$

Now the integrand in Eq. 5.8 can be rearranged as

$$\sqrt{|g|}(\partial_\mu \Lambda)\, j^\mu = \partial_\mu(\sqrt{|g|}\Lambda\, j^\mu) - \Lambda\, \partial_\mu \left(\sqrt{|g|} j^\mu)\right)$$

As before, the first term gives a boundary term and vanishes if $\Lambda j^\mu \to 0$ at infinity. $S_{int}$ is therefore guaranteed to be gauge invariance provided the current satisfies the continuity equation holds

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} j^\mu\right) = 0 \tag{5.11}$$

This is the covariant form of the continuity equation in curvilinear coordinates, taking into account that the current density is a density in the sense of tensors.

# Chapter 6

# The inhomogeneous Maxwell's equations

The inhomogeneous Maxwell equations are derived from a variational principle.

## 6.1 Lagrangian field theory

In Lagrangian mechanics the basic object is the Lagrangian, $L(q_j, \dot{q}_j, t)$, a function of the "generalized coordinates" $q_j$ and their velocities $\dot{q}_j$ and $j$ labels the degrees of freedom. Lagrangian field theory can be viewed as a generalization of Lagrangian mechanics to infinitely many degrees of freedom where the discrete index $j$ is replaced by $\mathbf{x}$, a point in space. The Lagrangian is then of the form $L = \int d\mathbf{x}\, \mathcal{L}_F$ where $\mathcal{L}_F$ is a suitable Lagrangian density for the field: a function of the fields and their time derivatives.

### 6.1.1 The Lagrangian of the electromagnetic field

Now we come to deciding what replaces the $q_j$ and $\dot{q}_j$ for the electromagnetic field. The first natural choice appears to be $F_{\mu\nu}$. However, $F$ can not be viewed as independent generalized coordinates, since they are constrained by the homogeneous Maxwell's equation. Independent generalized coordinates are $A_\mu$:

$$q_j \leftrightarrow A^\mu(x), \quad \dot{q}_j \leftrightarrow \dot{A}^\mu(x)$$

This reproduces Maxwell equations, as we shall see.

Lorentz invariance of the Euler-Lagrange equations is guaranteed if the action is a Lorentz scalar. We have at our disposal two Lorentz scalars[1] whose dimensions are energy density:

$$F \cdot F = F_{\mu\nu} F^{\mu\nu}, \quad F \cdot F^* = F_{\mu\nu} (F^*)^{\mu\nu},$$

[1] For simplicity we assume $|\det g| = 1$

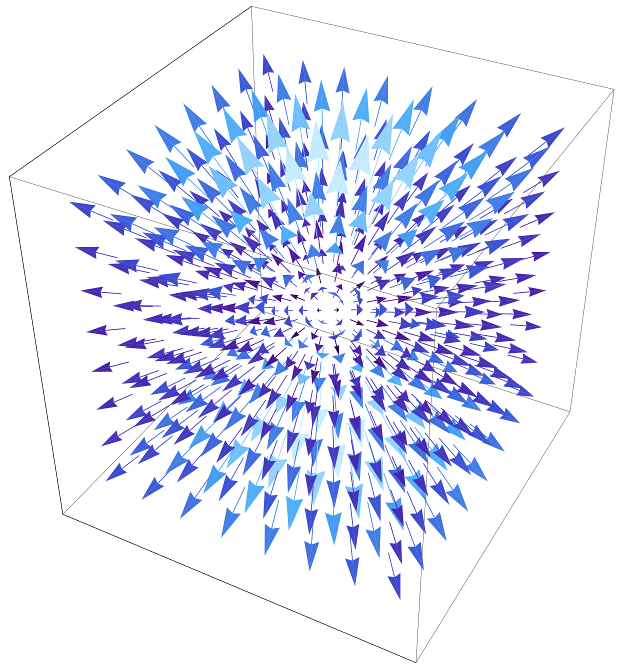


Figure 6.1: The action associated a number with a given field configuration and a box in space-time. We allow variation of $A^\mu$ inside the box: The variation vanishes outside the box and on its boundary. This is the analog of what we do when we vary the path.

Since the volume element in space-time $d\Omega = d^4x$ is a Lorentz scalar and since the action must[2] have dimension $[Et]$, two candidates for the field action $S_f$ are suitable numerical multiples of

$$S_F = -\frac{1}{16\pi c}\int \underbrace{d\Omega}_{volume\ element} F\cdot F, \quad S_{cs} = \frac{1}{c}\int d\Omega F^*\cdot F$$

We can rule out the action $S_{sc}$ by the following observation: The homogeneous Maxwell equation

$$\begin{aligned}(F^*)\cdot F &= 2(F^*)^{\mu\nu}(\partial_\mu A_\nu)\\ &= 2\partial_\mu\left((F^*)^{\mu\nu}A_\nu\right)\end{aligned}$$

This means that the associated action is a boundary term. Since the rules of variation keep the boundary terms fixed, the variation of $S_{cs}$ vanishes identically.

We are left with the first candidate. We need first to justify the sign chosen so that the action will have a minimum rather than a maximum.

[2] So we can add it to $S_p$ and $S_{int}$

In Lagrangian mechanics the kinetic energy comes with a positive sign. Since $\mathbf{E}$ is linear in $\dot{\mathbf{A}}$ while $\mathbf{B} = \nabla \times \mathbf{A}$, it is the $\mathbf{E}^2$ that plays the role of kinetic energy and as

$$F \cdot F = -2(\mathbf{E}^2 - \mathbf{B}^2) \tag{6.1}$$

we must have

$$S_F = -\frac{1}{16\pi c} \int F \cdot F \, d\Omega \tag{6.2}$$

The $16\pi$ gives Maxwell equations in c.g.s units and in particular leads to the Coulomb potential in the form[3] $\frac{e}{r}$. In the MKS system where Coulomb law is $e/4\pi\epsilon_0 r$ one needs to replace $16\pi$ by $4\epsilon_0$ .

## 6.2 Variation of the field: Rules of the game

The actions $S_F$ assigns a number for any given field $A_\mu(x)$. It is a functions whose arguments are the functions $A_\mu(x)$. Such objects are sometimes called functionals.

We consider variation $\delta S_F$ due to variations $\delta A_\mu$. We shall consider local variations only, namely, variations in a finite region of space time: $\delta A_\mu = 0$ outside some large space-time box, so we do not need to worry about infinite variations that can come with infinite boxes.

### 6.2.1 Variation of the field: Calculations

The variation of $A$ causes a variation of $F \cdot F$

$$\delta(F_{\mu\nu} F^{\mu\nu}) = 2F^{\mu\nu} \, \delta(F_{\mu\nu})$$

where

$$\delta(F_{\mu\nu}) = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu$$

By the anti-symmetry of $F$

$$\delta(F_{\mu\nu} F^{\mu\nu}) = 2F^{\mu\nu} \left(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu\right) = 4F^{\mu\nu} \left(\partial_\mu \delta A_\nu\right) \tag{6.3}$$

In the calculus of variations one wants to end up with an expression proportional to $\delta A_\mu$: We need to get rid of terms of the form $\delta \partial A$. This we can do by integrating by parts. In the case of curvilinear coordinates,

$$d\Omega = \sqrt{|g|} d^4 x \tag{6.4}$$

Since $\det |g|$ is a function of the coordinates, and not a function of $A$, it is not affected by the variation. From Eq. 6.3

$$\sqrt{|g|} \delta(F_{\mu\nu} F^{\mu\nu}) = 4\partial_\mu(\sqrt{|g|} F^{\mu\nu} \, \delta A_\nu) - 4\partial_\mu(\sqrt{|g|} F^{\mu\nu}) \, \delta A_\nu$$

[3] Replacing $16\pi$ by 4 in the action, gives for the Coulomb potential $e/4\pi r$.

The first term is the divergence of the vector field $\sqrt{|g|}F^{\mu\nu}\,\delta A_\nu$ which can be converted to a 4-surface integral on the boundary of the box where $\delta A = 0$. Hence,

$$\delta S_F = \frac{1}{4\pi c}\int d\Omega \underbrace{\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}F^{\mu\nu})}_{\text{divergence of anti-symmetric tensor}} \delta A_\nu \tag{6.5}$$

Maxwell's equations in free space follow from $\delta S_F = 0$ for any $\delta A_\nu$. This is the case if

$$\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}F^{\mu\nu}) = 0 \tag{6.6}$$

### 6.2.2 Variation of the interaction

We have already determined the action associated with the interaction when we studied the dynamics of relativistic charged particles as

$$S_{int} = \frac{1}{c^2}\int A_\nu j^\nu\, d\Omega$$

To get the field equations we consider variations $\delta A$ for a given source term $j$. This variation is

$$\delta S_{int} = \frac{1}{c^2}\int \delta A_\nu j^\nu\, d\Omega \tag{6.7}$$

### 6.2.3 The inhomogeneous Maxwell equations

The Euler-Lagrange equations for the fields are those that minimize the action $S_F + S_{int}$. The minimizer is the stationary point of the variation. When $\det|g| = 1$ we simply get

$$0 = \delta S_F + \delta S_{int} = \frac{1}{4\pi c}\int d\Omega \left(\partial_\mu F^{\mu\nu} + \frac{4\pi}{c}j^\nu\right)\delta A_\nu \tag{6.8}$$

The variation will vanish for arbitrary $\delta A$ provided the brackets vanish. In the case of curvilinear coordinates where $\det|g|$ is not necessarily 1 we have[4]

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}F^{\nu\mu}\right) = \frac{4\pi}{c}j^\nu \tag{6.9}$$

These are the 4-inhomogeneous Maxwell equations in a neat and concise form.

### 6.2.4 Current conservation

We have derived Maxwell equations as the Euler-Lagrange equations for the field $A_\mu$ for a given source term $j^\mu$. This derivation did not assume that the source $j^\mu$ is a reasonable physical current and did not explicitly require that it

[4]Note the interchange of the indices of $F$ relative to Eq. 6.8.

be a conserved current. However, a-posteriori, Maxwell equation enforce current conservation on $j$ as a direct consequence of the fact that $F$ is an antisymmetric tensor:

$$0 = \underbrace{\partial_{\mu\nu}\left(\sqrt{|g|}F^{\mu\nu}\right)}_{0 \ by \ symmetry} = \frac{4\pi}{c}\partial_\nu\left(\sqrt{|g|}j^\nu\right)$$

in accordance with Eq. 5.11. If the source $j$ was not current conserving, Maxwell equations would not form a consistent set of equation.

### 6.2.5 3-D form

To translate back Maxwell equations from their covariant space-time form to 3-D form, consider first the $\nu = 0$ equation. Since

$$E_j = F_{0j} = F^{j0}, \quad j^0 = c\rho$$

we get Gauss-Coulomb law

$$\nabla \cdot \mathbf{E} = 4\pi\rho \Longleftrightarrow \partial_\mu F^{0\mu} = \frac{4\pi}{c}j^0 \tag{6.10}$$

The spatial components are:

$$\partial_\mu F^{j\mu} = \partial_0 F^{j0} + \partial_k F^{jk} = -\frac{1}{c}\,\partial_t E_j - \partial_k(\varepsilon_{ikj}B_i) = \frac{4\pi}{c}j^j$$

Using

$$\partial_k(\varepsilon_{ikj}B_i) = -\varepsilon_{jki}\partial_k B_i = -(\nabla \times \mathbf{B})_j$$

This gives Ampere-Maxwell equation:

$$-\dot{\mathbf{E}} + \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} \Longleftrightarrow \partial_\mu F^{k\mu} = \frac{4\pi}{c}\,j^k \tag{6.11}$$

### 6.2.6 Time reversal

Time-reversal of the orbits of the sources $\xi_a(t) \mapsto \xi_a(-t)$, sends $\rho(t,\mathbf{x}) \mapsto \rho(-t,\mathbf{x})$ but flips the currents $\mathbf{J}(t,\mathbf{x}) \mapsto -\mathbf{J}(-t,\mathbf{x})$. It follows that solutions to Maxwell's equations transfrom under time-reversal as $\mathbf{E}(t,\mathbf{x}) \mapsto \mathbf{E}(-t,\mathbf{x})$ and $\mathbf{B}(t,\mathbf{x}) \mapsto -\mathbf{B}(-t,\mathbf{x})$. We say that $\mathbf{E}$ is even under time reversal and $\mathbf{B}$ is odd.

### 6.2.7 Maxwell equations: Evolution equations and constraints

Maxwell' equations for $\mathbf{E}$ and $\mathbf{B}$, partition into a set of two scalar equations and two vector equations. The two scalar equations are Gauss laws:

$$\underbrace{\nabla \cdot \mathbf{E} = 4\pi\rho}_{Gauss}, \quad \nabla \cdot \mathbf{B} = 0 \tag{6.12}$$

The two vector equations are Faraday and Maxwell-Ampere laws

$$\underbrace{\dot{\mathbf{E}} = \nabla \times \mathbf{B} - \frac{4\pi}{c}\mathbf{J}}_{Ampere}\,, \quad \underbrace{\dot{\mathbf{B}} = -\nabla \times \mathbf{E}}_{Faraday} \tag{6.13}$$

and dot denotes *partial* derivative with respect to $ct$. The vector equations are evolution equations that allow to propagate $\mathbf{E}$ and $\mathbf{B}$ in time, given their initial values and the source $\mathbf{J}$.

In total, there are 8 Maxwell equations for the 6 unknown fields. This looks like an over constrained system. It is better to view them as two evolution vector equation for two vectors and view the scalar equations as a constraint on the initial data. This constraint is preserved by the evolution provided $(\rho, \mathbf{J})$ satisfy the continuity equation.

**Exercise 6.1.** *Show that the evolution respects the constraint.*

**Example 6.2** (Current carrying wire)**.** *An electrically neutral, infinitely long, metallic straight wire (along the z-axis) with circular cross section of radius $a$ carries a stationary current $I$. Suppose Ohm's law in the form $\mathbf{J} = \sigma\mathbf{E}$ with $\sigma$ a constant in the wire. Find the profiles of the electric and magnetic fields inside and outside the wire. Assume cylindrical symmetry, translational symmetry in the z direction and stationarity. Analyze the problem in cylindrical coordinates $(\rho, \theta, z)$.*

*By Gauss and the symmetry $E_\rho = 0$ Since the magnetic field is assumed stationary, by Faraday and the symmetry $E_\theta = 0$. You might then be tempted to say that* outside the wire, *one should have $E_z = 0$. This, however, leads to a contradiction: Combinig Gauss and Faraday*

$$0 = \nabla \times \mathbf{E} \Rightarrow 0 = \nabla \times \big(\nabla \times \mathbf{E}\big) = -\Delta\mathbf{E} + \nabla\big(\nabla \cdot \mathbf{E}\big) \Rightarrow \Delta\mathbf{E} = 0$$

*Which says that $\mathbf{E}$ is harmonic everywhere. Hence, if it is zero outside the wire, it is zero everywhere. This, together with Ohm's law, contradicts the assumption that the wire carries current.*

*Let us then retreat to the next line of defense and take $\mathbf{E} = E_0\hat{\mathbf{z}}$ with $E_0$ a constant. This is still harmonic By the integral form of Ampere*

$$\mathbf{B} = \frac{2I}{c\rho}\hat{\theta} \times \begin{cases} 1 & \rho > a \\ \left(\frac{\rho}{a}\right)^2 & \rho < a \end{cases}$$

*where $I$ is the total current. We have used the fact that inside the wire, the constancy of $\mathbf{E}$ implies the constancy of $\mathbf{J}$.*

*It may be a little shocking at first that a neutral current carrying wire bundles with it an electric field that does not decay as you get far from the wire. This is a pathology due to the assumed infinite length of the wire.*

## 6.3 New Physics

Lagrangian field theory allowed us to repackage Maxwell's electrodynamics in an elegant formalism. But more importantly, it allows to explore new models. Most models are, at the end, just models, and with application as questions in homework sets and exams. But occasionally, some turn out to capture new physics.

## 6.4 Electrodynamics in 1+1 dimensions

It is not possible to explicitly solve Maxwell's equation for the fields for general motion of the sources. However, in 1+1 dimensions this is possible.

In 1+1 dimensions $F$ is an anti-symmetric $2 \times 2$ matrix. Its single entry is the electric field, which is a Lorentz scalar. Maxwell's equations are then

$$\partial_x E = 4\pi\rho, \quad \partial_t E = -4\pi J \tag{6.14}$$

and they are consistent if the source satisfies the continuity equation.

**Exercise 6.3** (Solution of Maxwell equations for arbitrary motion of a point charge)**.** *Show that for a charged particle with a given, arbitrary, orbit, the solution of Maxwell equations for $E$ takes two constant values in the space-time plane separated by the world line of the particle. Determine the jump across the world line.*

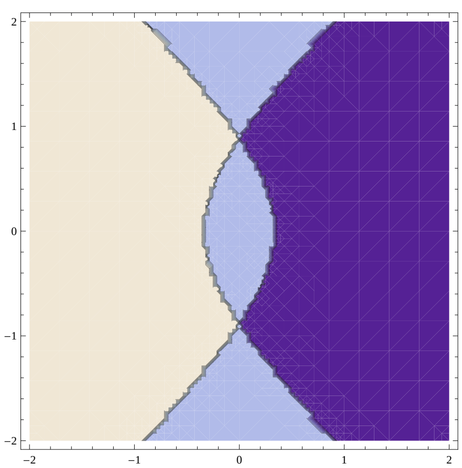



Figure 6.2: Maxwell's equation in 1+1 dimensions can be solved geometrically for arbitrary motion of the source. The figure illustrates the solution in space-time for two point sources undergoing constant acceleration. The field takes constant values in the different regions delineated by the orbits of the charges.

### 6.4.1 Axion

In Maxwell's theory the source term is a the vector field of currents $j^\mu$. In 1+1 dimensions there is a different option for a source term, namely a scalar field $\phi(x)$ [5] :

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\phi(x)\varepsilon^{\mu\nu}F_{\mu\nu}$$

This looks first like a different theory, but it is actually equivalent to Maxwell's. The variation of $A$ gives, up to boundary terms,

$$\delta\mathcal{L} = (\partial_\mu F^{\mu\nu})\delta A_\nu - \partial_\mu\phi(x)\varepsilon^{\mu\nu}\delta A_\nu$$

The Euler-Lagrange equations for this model are then

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad j^\nu = (\partial_\mu\phi)\varepsilon^{\mu\nu} \tag{6.15}$$

Note that $\partial_\nu j^\mu = 0$ so the current is conserved. We have recovered Maxwell theory except that the current is interpreted as the gradient of a scalar.

## 6.5 The quantum Hall effect

The quantum Hall effect was discovered in the 1980's ushered a new era of research now called the study of *topological phases*. These phases are are intrinsically quantum and are topolgical in the sense that the quantum state of the system has certain topological features that I shall not go into. The Hall effect is a two dimensional phenomenon and the topological phases are labeled a quantized value of the Hall conductance. Let me explain.

In 2 dimensions the conductance is a rank 2 tensor. There are two rank 2 tensors that are rotationally invariant: The identity and Levi-Civita. So the most general isotropic conductance is

$$\mathbf{J} = \begin{pmatrix} \sigma_\parallel & \sigma_H \\ -\sigma_H & \sigma_\parallel \end{pmatrix}\mathbf{E}, \quad \sigma > 0 \tag{6.16}$$

The diagonal part is the dissipative conductance, and this is why it is always positive. The off-diagonal is the Hall conductance and can have either sign.

At sufficiently low temperatures, and sufficiently strong magnetic field, the system is almost in the ground state. One then finds that $\sigma_H$ as a function of $B$ displays the following features

- It is a step-like function of $B$
- In the steps, there is no dissipation and the Hall conductance is quantized

$$\sigma_\parallel = 0, \quad \sigma_H \in \mathbb{Q}\frac{e^2}{h} \tag{6.17}$$

  $\mathbb{Q}$ denotes the rationals. Planck constant is an indication that the phenomenon is quantum.

[5] Frank Wilczek, who invented this field, called it Axion field.

- The magnetic field and the charge density are related by

$$\partial_B(c\rho) = \sigma_H \tag{6.18}$$

We can summerize the equations 6.16-6.18 in one space-time vector equation

$$j = \sigma_H F^*, \quad j = (c\rho, j^1, j^2), \quad F^* = (B, E_2, -E_1) \tag{6.19}$$

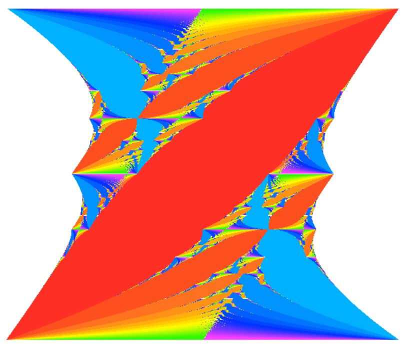


Figure 6.3: The phase diagram for the Integer quantum Hall effect for the Hofstadter model on the triangular lattice at $T = 0$. The vertical axis is the magnetic flux through the unit cell. The horizontal axis is the chemical potential. Figure made by Gal Yehoshua for an undergrad project.

### 6.5.1 The Chern-Simons action

The Chern Simons field theory captures some of the basic features of the quantum Hall effect:

- It is an intrinsically 2 space dimensional theory
- It has broken time reversal and space inversion, symmetries that are broken by the large external magnetic field
- It incorporates charge conservation.
- It has the right scaling dimension

Charge coservation in 2+1 dimensions is

$$0 = \partial_\mu j^\mu, \quad j = (c\rho, j^1, j^2) \tag{6.20}$$

If you think of current conservation as the statement that the divergence of a vector field vanishes, then this vector field must be the rotation of some vector field. To make contact with the Hall effect we identify the vector field with the electromagnetic gauge field $A$

$$j^\alpha = k\epsilon^{\alpha\beta\gamma}\partial_\beta A_\gamma \tag{6.21}$$

$k$ is an appropriate constant that we shall choose later[6]. It is easy to see that Eq. 6.20 indeed holds. Since we picked $A$ to be the electromagnetic gauge field, by definition of the dual

$$(F^*)^\alpha = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}\partial_\beta A_\gamma \tag{6.22}$$

We can now ask what Lagrangian would give Eq. 6.19 as its Euler Lagrange equations ?

Since Eq. 6.19 is Lorentz covariant, the Lagrangian must be a Lorentz scalar. As the Hall effect breaks time and space reflections, we want the Lagrangian to have this feature as well. The Levi-Civita tensor breaks these symmetries, so the Lagrangian should be the contraction of Levi-Civita $\varepsilon^{\alpha\beta\gamma}$ with a third rank tensor made from $F$ and $A$. There is one way to do that, and this is the Chern-Simons action

$$\mathcal{L}_{CS} = \frac{k}{4\pi c^2}(F^*)^\alpha A_\alpha = \frac{k}{8\pi c^2}\varepsilon^{\alpha\beta\gamma}F_{\beta\gamma}A_\alpha, \tag{6.23}$$

$\alpha$ runs over $0, 1, 2$. $k$ is a constant that we shall adjust later.

The Chern Simons Lagrangian density is not gauge invariant. However, it nevertheless leads to gauge invariant equations of motion. Under a change of gauge $A_\mu \to A_\mu + \partial_\mu\Lambda$, the Chern-Simons Lagrangian density $\mathcal{L}_{CS}$ changes by a boundary term.

$$\mathcal{L}_{CS} \mapsto \mathcal{L}_{CS} + \frac{k}{8\pi c^2}\,\varepsilon^{\alpha\beta\gamma}F_{\beta\gamma}\partial_\alpha\Lambda = \mathcal{L}_{CS} + \frac{k}{4\pi c^2}\,\varepsilon^{\alpha\beta\gamma}\partial_\alpha\Big(\partial_\beta A_\gamma\Lambda\Big) \tag{6.24}$$

**Exercise 6.4.** *Show that $\partial_\mu(F^*)^\mu = 0$ is a consequence of the fact that $F$ is defined by the potentials $A$. (This is known as Bianchi identity.) Interpret Bianchi as the equations in terms of Faraday law.*

The CS Lagrangian is that it breaks both time reversal symmetry and parity. You can see this either from the fact that it is first order in the derivatives, or from the Levi-Civita tensor. This is a reflection of the symmetry of the Hall effect where the external magnetic field breaks both symmetries.

To find the equations of motion for CS consider first the variation of the action

$$\begin{aligned}\delta(F^*A) &= \varepsilon^{\alpha\beta\gamma}(\delta\partial_\beta A_\gamma)A_\alpha + (F^*)^\alpha\delta A_\alpha \\ &= -\varepsilon^{\alpha\beta\gamma}(\delta A_\gamma)(\partial_\beta A_\alpha) + (F^*)^\alpha\delta A_\alpha + \partial_\beta(\varepsilon^{\alpha\beta\gamma}A_\alpha\delta A_\gamma) \\ &= 2(F^*)^\alpha\delta A_\alpha + \partial_\beta(\dots)\end{aligned}$$

[6] By dimensional analysis $k$ has the dimensions of conductance.

The interaction term is the same as in Maxwell theory, and the full Lagrangian is

$$\mathcal{L}_{CS} + \frac{1}{c^2} A_\alpha j^\alpha$$

The Euler-Lagrange equations are

$$\frac{k}{2\pi} F^* + j = 0 \tag{6.25}$$

Unlike Maxwell's equations, this is not a set of differential equation, but an algebraic relation between the fields and sources. Comparing with Eqs. 6.19 gives

$$k = -2\pi\sigma_H \tag{6.26}$$

### CS is the boundary of axion electrodynamics in the bulk

CS can be viewed as the holographic shadow of the action associated with $\mathbf{E} \cdot \mathbf{B}$ in the bulk:

$$\begin{aligned} \tfrac{1}{4} F_{\mu\nu} (F^*)^{\mu\nu} &= \varepsilon^{\alpha\beta\gamma\delta} \partial_\alpha A_\beta \partial_\gamma A_\delta \\ &= \partial_\alpha \left( \varepsilon^{\alpha\beta\gamma\delta} A_\beta \partial_\gamma A_\delta \right) \qquad (6.27) \\ &= \tfrac{1}{2} \partial_\alpha \left( \varepsilon^{\alpha\beta\gamma\delta} A_\beta F_{\gamma\delta} \right) \qquad (6.28) \end{aligned}$$

It follows that

$$\tfrac{1}{2} \int dx^3 F_{\mu\nu} F^{\mu\nu} = \int dS_\alpha \varepsilon^{\alpha\beta\gamma} A_\beta F_{\gamma\delta} \tag{6.29}$$

CS is therefore related to the 3+1 axion electrodynamics associated with the 2+1 dimensional boundary.

### Quantization

To explain why $k$ must be quantized one needs to input quantum mechanics and also assume that the two dimensional space of the system is a closed mmanifold, e.g. a two dimensional torus.

In quantum mechanics one allows the system to explore all configurations. A configuration is weighted by a complex phase

$$e^{iS/\hbar}$$

The CS action changes by a boundary term under a gauge transformation. The weight therefore is gauge invariant only if under a gauge transformation $\Lambda$

$$\frac{k}{4\pi\hbar c^2} \int_{bdry} F^*_\alpha \Lambda dS^\alpha = 0 \quad Mod\ 2\pi \tag{6.30}$$

To examine this condition, we need to discuss how $\Lambda$ enters into quantum mechanics. $\Lambda$ affects the phase of the wave function. Since $[\Lambda] = [e]$ and phase is dimensionless, we first need to adjust the dimensions. As $\alpha = e^2/\hbar c$ is dimensionless, a dimensionless quantity is

$$\frac{\alpha^n}{e}\Lambda$$

The phase should have something to do with QM the simplest choice for a phase change is

$$e^{ie\Lambda/\hbar c} \tag{6.31}$$

The next interesting thing to observe about this expression is that a constant $\Lambda$ of the form

$$e^{ie\Lambda/\hbar c} = 1 \tag{6.32}$$

is a trivial gauge transformation. Therefore, a gauge transformation

$$\Lambda(t) = \frac{hc}{2e}(\tanh t/T + 1) \tag{6.33}$$

is asymptotically trivial: It does not affect either $A_\mu$ or the phase of the wave function in the distant past and the distant future. The condition that the weight of a path is gauge invariant, Eq. 6.30 is then

$$0 \quad Mod\ 2\pi = \frac{k}{4\pi\hbar c^2}\int_{bdry} F^*_\alpha \Lambda dS^\alpha \tag{6.34}$$

Now suppose the physical system is a torus with $-L < x, y < L$ and that the time is $-10T < t < 10T$. The boundary of this 3-D box has 6 faces, so the integral above can be broken into 6 integrals

$$\int \Lambda F^*_t dxdy\Big|_{t=\pm 10T} + \underbrace{\int \Lambda F^*_t dydx^0\Big|_{x=\pm L}}_{=0} + \underbrace{\int \Lambda F^*_t dx^0 dx\Big|_{y=\pm L}}_{=0} \tag{6.35}$$

The terms that vanish do so because $\Lambda F^*$ is a periodic function in $x$ and $y$ with period $2L$. Since $\Lambda \approx 0$ for $t = -10T$ and $\Lambda \approx hc/e$ for $t = 10T$ we finally get the quantization condition

$$0 \quad Mod\ 2\pi = \frac{k}{4\pi\hbar c^2}\int F^*_0 \Lambda dxdy = \frac{k}{4\pi ce}\int Bdxdy \tag{6.36}$$

Now we need one more input from QM: Dirac monopole condition. In the case that the Hall system is a 2-D torus, the magnetic flux through the torus is quantized

$$\frac{e}{\hbar c}\int B\ dxdy = 2\pi m, \quad m \in \mathbb{Z}$$

This gives for the rhs of Eq. 6.34

$$0 \quad \mathrm{mod}\ 2\pi = \frac{k}{4\pi ce}\int Bdxdy = \frac{mk\hbar}{2e^2} \tag{6.37}$$

which quantizes $mk$ to a multiple of $e^2/\hbar$. Since $k = -2\pi\sigma_H$ we get that the Hall conductance is a fraction multiple of the quantum unit of conductance

$$\sigma_H = \frac{e^2}{h}\frac{2N}{m}, \quad N, m \in \mathbb{Z} \tag{6.38}$$

## 6.6 Supplement: Axion electrodynamics

In 3+1 dimensions $F^* \cdot F$ is a boundary term, and as such it does not affect the equations of motion. However, this terms can do something interesting if its coupling constant is replaced by a function. The function is called the Axion field $\phi(x)$:

$$\mathcal{L} = -\frac{1}{16\pi c}\, F \cdot F - \frac{\sigma_0}{8\pi}\, \phi(x) F^* \cdot F$$

Evidently, this Lagrangian is gauge invariant[7]. Since

$$\phi F^* \cdot F = 2\phi\, (F^*)^{\mu\nu} \partial_\mu A_n = 2\phi \partial_\mu \big((F^*)^{\mu\nu} A_\nu\big) = -2(\partial_\mu \phi)\, (F^*)^{\mu\nu} A_\nu + \partial_\mu(\dots)$$

we can then replace it by (a formally gauge dependent Lagrangian density)

$$\mathcal{L} = -\frac{1}{16\pi c} F \cdot F - \frac{\sigma_0}{4\pi} (\partial_\mu \phi)(F^*)^{\mu\nu} A_\nu$$

Up to boundary terms, the variation of the action is

$$4\pi c\, \delta\mathcal{L} = (\partial_\mu F^{\mu\nu})\, \delta A_\nu + \alpha\, (\partial_\mu \phi)(F^*)^{\mu\nu} \delta A_\nu,$$

**Exercise 6.5.** *Determine* $\alpha$ (Answer:$\alpha = -3\frac{e^2}{\hbar c}$)

The Euler-Lagrange equations are

$$(\partial_\mu F^{\mu\nu}) = -\alpha\, (\partial_\mu \phi)(F^*)^{\mu\nu}$$

It is instructive to write the equations in terms of **E** and **B**. Gauss law (without external sources) now takes the form

$$\nabla \cdot \mathbf{E} = \alpha (\nabla \phi) \cdot \mathbf{B}$$

Ampere law

$$\partial_0 \mathbf{E} - \nabla \times \mathbf{B} = \alpha \dot{\phi} \mathbf{B} + \alpha \nabla \phi \times \mathbf{E}$$

**Exercise 6.6.** *Verify.*

When $\phi$ is a constant one recovers the sourceless Maxwell equations. In general, $\partial_\mu \phi$ acts like a source term in Maxwell equations.

### 6.6.1 Quantum interface

Axion electrodynamics started as a speculative model of an elementary particle: The Axion. A different perspective was taken by Qi et. al. who proposed looking at the interface between topologically distinct quantum phases. In the bulk of the two insulators Maxwell theory applies. This says that $\phi$ is constant in each. The constant is quantized to be 0 or $\pi$, by an gauge invariance argument

[7] Since **E** is even and **B** odd under time reversal the Lagrangian breaks time-reversal unless $\phi$ is also odd under time reversal. The notion of time reversal in the quantum case is subtle.

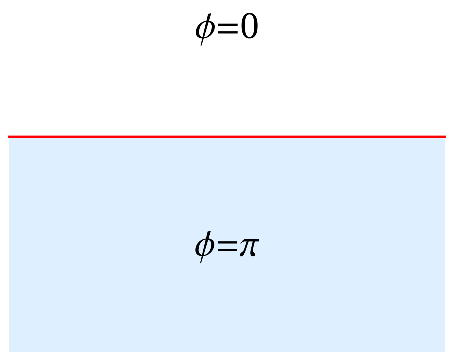



Figure 6.4: The interface between two insulators that are topologically distinct gives rise to a singular Axion field.

similar to the one in CS theory of the quantum Hall effect. By definition, the two insulators are topologically distinct if the constant is different in each. This means that $\nabla\phi = \delta^{(2)}(\mathbf{x})\mathbf{n}$. Gauss law is replaced by

$$\nabla \cdot \mathbf{E} = \alpha\delta^{(2}(\mathbf{x})\mathbf{n} \cdot \mathbf{B}$$

The magnetic field on the surface acts as if there was a charge on the interface. This is something we have already encountered in the CS theory of the quantum Hall effect.

Ampere law is replaced by

$$\partial_0\mathbf{E} - \nabla \times \mathbf{B} = \alpha\delta^{(2)}(\mathbf{x})\mathbf{n} \times \mathbf{E}$$

This means that electric field on the surface acts as if there were currents at the interface. In particular in Axion electro-Magneto-statics

$$\nabla \cdot \mathbf{E} = \alpha\delta^{(2}(\mathbf{x})\mathbf{n} \cdot \mathbf{B}, \quad -\nabla \times \mathbf{B} = \alpha\delta^{(2)}(\mathbf{x})\mathbf{n} \times \mathbf{E}$$

### 6.6.2 Magnetic response to an electric field

Consider a non-trivial insulator in the form of an infinitely long cylinder of radius $a$ immersed in the trivial vacuum and z-oriented. Take uniform electric field everywhere and uniform magnetic field inside the cylinder

$$\mathbf{E} = E_0\hat{\mathbf{z}}, \quad \mathbf{B} = B_0\,\theta(a-\rho)\,\hat{\mathbf{z}}$$

Clearly

$$\nabla \cdot \mathbf{E} = 0, \quad \mathbf{n} \cdot \mathbf{B} = 0$$

so Axion Gauss law is satisfied.

$\nabla \times \mathbf{B}$ vanishes inside the cylinder and outside the cylinder, but has a delta jump on the boundary. The magnetic field in the wire is proportional to the constant electric field:

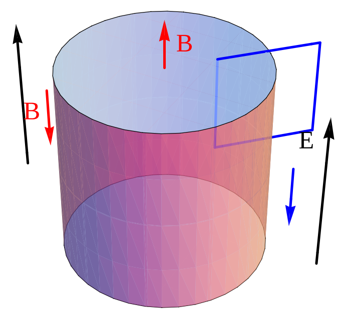



Figure 6.5: A cylinder of a (non-trivial) topological insulator is immersed in vacuum (trivial insulator). A uniform electric field $\mathbf{E}$ in the axial direction leads to response in a magnetic field inside the cylinder

**Exercise 6.7.** *Use Stokes theorem for the (blue) rectangle shown in the figure to show that*

$$B_0 = \alpha E_0$$

### 6.6.3 Phantom monopoles

In electrostatics an electric charge near a (grounded) conductor has an oppositely charged image. I want to show that in Axion electrodynamics you can create image which is a magnetic monopole. We want to find consistent solution as if there was a magnetic monopole in the lower half space. Namely

$$\mathbf{B}(\mathbf{x}) = g\frac{\mathbf{x}+d\hat{\mathbf{z}}}{|\mathbf{x}+d\hat{\mathbf{z}}|^3}\,\theta(z) + (yet\ unknown\ function)\theta(-z)$$

**Exercise 6.8.** *Explain why* $\nabla\cdot\mathbf{B} = \nabla\times\mathbf{B} = 0$ *in the half space* $z > 0$

The magnetic provides a source term for the electric field. The source term is precisely the same as the source term in the corresponding electrostatic image charge problem provided

$$g\alpha = 2e$$

Now, if we add to this field the electric field given by electric monopole of charge $e$ above the x-y plane we obtain the same electric field configuration as in the electrostatic image charge problem, everywhere, i.e.

$$\mathbf{E} = e\left(\frac{\mathbf{x}-d\hat{\mathbf{z}}}{|\mathbf{x}-d\hat{\mathbf{z}}|^3} - \frac{\mathbf{x}+d\hat{\mathbf{z}}}{|\mathbf{x}+d\hat{\mathbf{z}}|^3}\right)\theta(z)$$

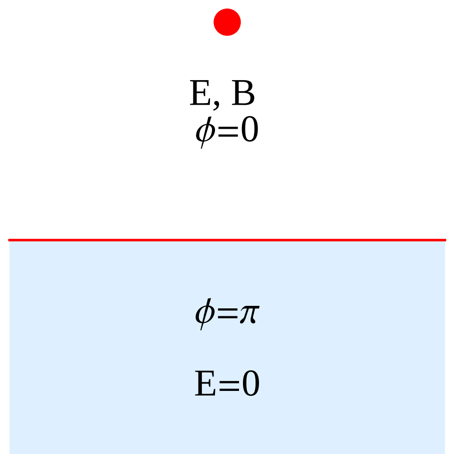



Figure 6.6: A electric charge, (red dot) is placed near a different topological insulator with zero fields. On the left, the physical setup. On the right the image method.

This describes the electric field everywhere. It remains to see what values **B** takes in the lower half-space. Now $\mathbf{E}\cdot\mathbf{n} = 0$ on the boundary and so we see that **B** is the solution of

$$\nabla \times \mathbf{B} = \nabla \cdot \mathbf{B} = 0$$

everywhere subject to the boundary condition that fixes **B** on the plane $z = 0$.

We introduce a scalar potential for **B** in the lower half pace

$$\mathbf{B} = \nabla\phi, \quad \Delta\phi = 0$$

subject to the boundary condition $\nabla\phi = \mathbf{B}$ on the plane $z = 0$. Evidently

$$\phi(x, y, z = 0) = \frac{g}{\sqrt{x^2 + y^2 + d^2}}, \quad B_z = \partial_z\phi = g\frac{d}{|x^2 + y^2 + d^2|^{3/2}}$$

The problem then reduces to solving Laplace equation with two types of boundary conditions.

**Bibliography** Xiao-Liang Qi, et al, Inducing a Magnetic Monopole with Topological Surface States, Science 323, 1184 (2009);

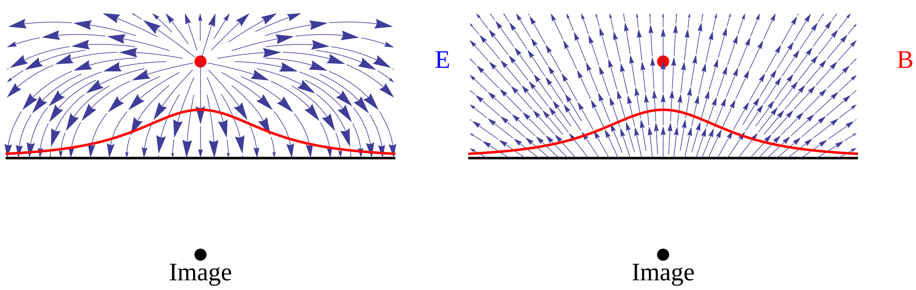



Figure 6.7: The red curve shows the surface charge density that allows the field to terminate at the surface. On the right one sees the response in the form of a magnetic field that seems to have a magnetic monopole at the image point. There is no real magnetic monopole anywhere, of course.

# Chapter 7

# Magnetic fields and magnetic induction

So far we considered the electric field and the magnetic induction $(\mathbf{E}, \mathbf{B})$, and derived Maxwell equations from Lorentz invariance. Let us now turn to $(\mathbf{D}, \mathbf{H})$ known as the displacement field and magnetic field.

One good reason to introduce two different notions of electric fields, $(\mathbf{E}, \mathbf{D})$ and two different notions of magnetic fields $(\mathbf{B}, \mathbf{H})$ is that they are associated with a-priory different measurements. $(\mathbf{E}, \mathbf{B})$ are defined by measuring the force and then using Coulomb-Lorentz law to determine $(\mathbf{E}, \mathbf{B})$. $\mathbf{D}$ is defined by measuring the charge on a pair of metallic plates, as in Fig. 7.1 and $H$ by measuring the surface current on a thin superconductor, as in Fig. 7.2.

A second good reason is that they are associated with different equations: $(\mathbf{E}, \mathbf{B})$ are associated with the homogeneous Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0 \tag{7.1}$$

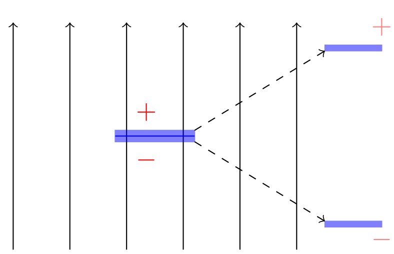



Figure 7.1: Put two thin metallic plates in contact in the field. The surface charge density on the plates is proportional to field strength. Separate the plates and measure the charge on the top plate. Define the field $D$ as the maximal charge per unit area over all initial orientations of the plates.

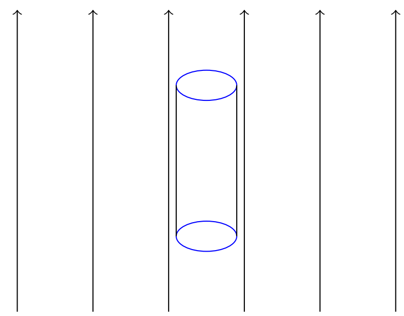


Figure 7.2: A thin superconducting cylinder expels the magnetic field by creating surface currents. Measuring the current gives $H$.

while $(\mathbf{D},\mathbf{H})$ are associated with the inhomogeneous Maxwell equations

$$\nabla\cdot\mathbf{D} = 4\pi\rho, \quad -\dot{\mathbf{D}} + \nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J} \tag{7.2}$$

In a stationary case $\mathbf{D} = 0$ inside a metal and has vanishing tangential component, $\mathbf{D}_\parallel = 0$, on the surface. Hence by Eq. 7.2, on the surface charge on a metallic surface $D_\perp$ is, up to a factor $4\pi$, proportional to the surface charge density. This explains why measuring the charges on the plates in Fig. 7.1 is a measure of $D$. Similarly, in a stationary case $\mathbf{H} = 0$ inside a superconductor, and $\mathbf{H}_\parallel$ is proportional to the surface current. This explains why measuring the current in Fig. 7.2 is a measurement of $\mathbf{H} = 0$.

The in-homogeneous equations follow from charge conservation: Given $\rho$, define $\mathbf{D}$ by the solution of the differential equation

$$\nabla\cdot\mathbf{D} = 4\pi\rho \tag{7.3}$$

The solution is explicitly given by an integral: By the superposition principle

$$\mathbf{D}(\mathbf{x},t) = \int d^3y\,\rho(\mathbf{y},t)\,\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3} \tag{7.4}$$

Now, use this Eq. 7.3 in the equation for charge conservation

$$0 = \partial_t\rho + \nabla\cdot\mathbf{J} = \frac{c}{4\pi}\nabla\cdot\left(\dot{\mathbf{D}} + \frac{4\pi}{c}\mathbf{J}\right) \tag{7.5}$$

Since the brackets have zero divergence it is the a curl of something. This something is $\mathbf{H}$

$$\dot{\mathbf{D}} + \frac{4\pi}{c}\mathbf{J} = \nabla\times\mathbf{H} \tag{7.6}$$

This is the inhomogeneous Maxwell's equations.

We can amalgamate the two fields in tensors. **E** and **B** in $F$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B^z & -B^y \\ E_y & -B^z & 0 & {}^x \\ E_z & B^y & -B^x & 0 \end{pmatrix} \tag{7.7}$$

and **D** and **H** in $\mathcal{D}$

$$\mathcal{D}^{\mu\nu} = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{pmatrix} \tag{7.8}$$

## 7.1 Constitutive relations

Maxwell equations written in terms of $F$ and $\mathcal{D}$ are

$$\partial_\mu (F^*)^{\mu\nu} = 0, \quad \partial_\mu \mathcal{D}^{\mu\nu} = \frac{4\pi}{c} j^\mu \tag{7.9}$$

To close the set we need a constitutive relation between $F$ and $\mathcal{D}$. The general form of such a relation is

$$\mathcal{D}^{\mu\nu} = \varepsilon_0^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{7.10}$$

$\varepsilon_0^{\mu\nu\alpha\beta}$ is the permittivity (not to be confused with Levi-Civita). It is a tensor of rank 4, which is anti-symmetric in the first pair of indices and the last pair and so has, in principle, 36 components. It is a property of the material. In the rest frame of the material it is a function of position, reflecting the composition of the material, but not a function of time if the material is in equilibrium. It is a generalization of the familiar relation

$$D_j = (\varepsilon_0)_{jk} E^k, \quad B^j = \mu^{jk} H_k \tag{7.11}$$

describing the relation between the electric displacement field and the electric field in a dielectric and between the magnetic induction and the magnetic field in a magnetic materials. Under Lorentz transformations $\varepsilon_0$ transforms like a 4-th rank tensor. This reflects the fact that a constitutive relation normally breaks Lorentz invariance since it is a property of a medium that has a rest frame.

In the case of free space the constitutive relation is the same in all Lorentz frames. Indeed, since in vacuum $F^{\mu\nu} = \mathcal{D}^{\mu\nu}$ we have

$$\mathcal{D}^{\mu\nu} = \varepsilon_0^{\mu\nu\alpha\beta} F_{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta} \tag{7.12}$$

It follows that

$$\varepsilon_0^{\mu\nu\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\nu\alpha}\eta^{\mu\beta}\right) \tag{7.13}$$

and we wrote $\varepsilon_0$ so that it is explicitly unti-symmetric in the first and last pair of indices. The tensor $\varepsilon_0$ plays the role of the identity for 4-th rank anti-symmetric tensors. It is invariant under Lorentz transformations, since $\eta$ is.

This relation for **E** and **D** takes the form

$$E^j = D^j = (\varepsilon_0)^{jk} E_k = \eta^{jk} E_k \tag{7.14}$$

This gives the Minkowski metric plays the interpretation of a dielectric constant of the vacuum. This observation is the starting point of the theory of cloaking.

## 7.2 Poarization and Magnetization

When we consider Maxwell's equation near or inside material bodies, it is the sources can be partitioned as

$$\rho = \rho_{micro} + \rho_{macro}, \quad \mathbf{J} = \mathbf{J}_{micro} + \mathbf{J}_{macro} \tag{7.15}$$

we normally can not describe the microscopic sources.

The homogeneous Maxwell are oblivious to the sources and retain their usual form

$$\nabla \cdot \mathbf{B} = 0, \quad \dot{\mathbf{B}} + \nabla \times \mathbf{E} = 0 \tag{7.16}$$

The inhomogeneous equation care about the sources, so if we want to get rid of the microscopic sources we need to do something. We do that by replacing the microscopic sources by fields that describe the medium: The polarization **P** and the magnetization **M**. Both are a property of the microscopic sources and so are confined to the material bodies containing them:

$$\mathbf{P}(\mathbf{x}) = \mathbf{M}(\mathbf{x}) = 0, \quad \mathbf{x} \in \{\text{outside body}\} \tag{7.17}$$

The polarization and magnetization characterizes the microscopic source by

$$\rho_{micro} = -\nabla \cdot \mathbf{P} =, \quad \frac{1}{c}\mathbf{J}_{micro} = \dot{\mathbf{P}} + \nabla \times \mathbf{M} \tag{7.18}$$

The inhomogeneous Maxwell equations involving all the sources are

$$\nabla \cdot \mathbf{E} = 4\pi(\rho_{micro} + \rho_{macro}), \quad -\dot{\mathbf{E}} + \nabla \times \mathbf{B} = \frac{4\pi}{c}(\mathbf{J}_{micro} + \mathbf{J}_{macro}) \tag{7.19}$$

If we define

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$$

we find differential equations that involve only the macroscopic sources:

$$\nabla \cdot \mathbf{D} = 4\pi\rho_{mac}, \quad -\dot{\mathbf{D}} + \nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{mac} \tag{7.20}$$

We succeeded in getting rid of the microscopic sources at the price of doubling the number of unknown fields to four vector fields: $(\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H})$. It is instructive to go again through the counting of equations and unknown fields:

- The homogenoues Maxwell equations Eq. 7.16 for the six fields $(\mathbf{E}, \mathbf{B})$ should be thought of as one vector valued evolution for $\dot{\mathbf{B}}$ and a constraint on the initial data.
- The in-homogenoues Maxwell equations Eq. 7.20 for the six fields $(\mathbf{D}, \mathbf{H})$ should be thought of as one vector valued evolution for $\dot{\mathbf{D}}$ and a constraint on the source term: The continuity equation.
- The missing 6 equations are the constitutive relations

$$\mathcal{D}^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{7.21}$$

  where the tensor $\varepsilon$ characterized the material. For the vacuum $\varepsilon$ is given by Eq. 7.13.
- In the general $\varepsilon$ must be viewed, in general, as a tensorial linear operator. If the material is homogeneous and translation invariant, then a translation invariant linear operator is a convolutions in space time. In the case that the material is also memory-less and has zero-range correlations, $\varepsilon$ reduces to a constant. This is the case for the vacuum[1].

Since $\mathbf{D}, \mathbf{E}$ are even under time reversal while $\mathbf{B}, \mathbf{H}$ are odd, when time-reversal is a symmetry the constitutive relation takes the form

$$D^j = \varepsilon^{jk} E_k, \quad B_j = \mu_{jk} H^k \tag{7.22}$$

In the case of isotropic media $\varepsilon^{jl} = \varepsilon_0 \delta^{jk}$ and $\mu_{jk} = \mu_0 \delta_{jk}$ are proportional to the identity.

**Exercise 7.1.** *Show that the microscopic charge distribution of a homogeneously polarized sphere of radius $a$ is concentrated on the surface with surface density*

$$\mathbf{P} \cdot \hat{\mathbf{x}} \; \delta(|\mathbf{x}| - a)$$

**Exercise 7.2.** *Show that the current distribution of a homogeneously magnetized sphere of radius $a$ is concentrated on the surface with surface density*

$$\mathbf{M} \times \hat{\mathbf{x}} \; \delta(|\mathbf{x}| - a)$$

**Exercise 7.3.** *Find $\mathbf{E}, \mathbf{D}$ inside and outside a uniformly polarized sphere.*

[1] This has to be revisited in a quantum thery.

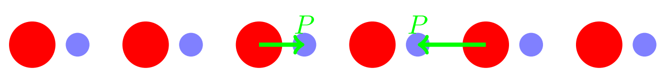



Figure 7.3: The intuitive notion of polarization of infinite macroscopic bodies in terms of dipole moments is confusing: Is the polarization the short arrow pointing to the right or the long arrow pointing to the left? Only by looking at the boundary on can tell.

# Chapter 8

# Cloaking

## 8.1 Dielectric media

In a dielectric medium, the homogeneous equations, Faraday's law and no-monoples, are the same as in free space[1]. In curvilinear coordinates they take the form

$$\nabla \cdot \mathbf{B} = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}B^j) = 0, \quad \dot{B}^j + \frac{\varepsilon^{jk\ell}}{\sqrt{g}}\partial_k E_\ell = 0 \qquad (homogenous)$$

Dot stands for derivative with respect to $x^0$. In the absence of external sources, the inhomogeneous Maxwell equations, Gauss and Ampere laws, in a dielectric are

$$\nabla \cdot \mathbf{D} = \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}D^j) = 0, \quad \dot{D}^j - \frac{\varepsilon^{jk\ell}}{\sqrt{g}}\partial_k H_\ell = 0 \qquad (inhomogeneous)$$

There are twelve unknown fields, and six evolution equations and two constraints. The missing equations are the constitutive relations[2]

$$D^j = \varepsilon^{jk}E_k, \quad B^j = \mu^{jk}H_k$$

where $\varepsilon$ and $\mu$ are tensors. We can then write for Maxwell equations for dielectrics, in the absence of sources as equations for $\mathbf{E}, \mathbf{H}$

$$\partial_j(\sqrt{g}\mu^{jk}H_k) = 0, \quad \mu^{jk}\dot{H}_k + \frac{\varepsilon^{jk\ell}}{\sqrt{g}}\partial_k E_\ell = 0 \tag{8.1}$$

and

$$\partial_j(\sqrt{g}\varepsilon^{jk}E_k) = 0, \quad \varepsilon^{jk}\dot{E}_k - \frac{\varepsilon^{jk\ell}}{\sqrt{g}}\partial_k H_\ell = 0 \tag{8.2}$$

[1]In Landau Lifshitz, **E** and **B** represent averages over a macroscopical small, but microscopically large, ball.

[2]Hopefully no confusion will arise between $\varepsilon$ as dielectric constant and $\varepsilon$ as Levi-Civita tensor.

The vacuum may then be thought of as a dielectric where the metric tensor is the dielectric constant, in c.g.s, is:

$$\varepsilon = \mu = g$$

where $g$ is the (Euclidean) metric. In a Euclidean metric $g$ is proportional to the identity and in any metric ${g^j}_k = \delta^j_k$. For example, in a two dimensional space described by cylindrical coordinates the covariant components of the metric and contravariant components of the dielectric tensors are

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}, \quad \varepsilon = \mu = \begin{pmatrix} 1 & 0 \\ 0 & 1/\rho^2 \end{pmatrix} \tag{8.3}$$

In practice, it is often the case that $\mu \approx g$. Then Maxwell's equations take the (approximate) form

$$\partial_j(\sqrt{g}g^{jk}H_k) = 0, \quad \sqrt{g}g^{jk}\dot{H}_k + \varepsilon^{jk\ell}\partial_k E_\ell = 0 \tag{8.4}$$

and

$$\partial_j(\sqrt{g}\varepsilon^{jk}E_k) = 0, \quad \sqrt{g}\varepsilon^{jk}\dot{E}_k - \varepsilon^{jk\ell}\partial_k H_\ell = 0 \tag{8.5}$$

## 8.2 Invisible dielectrics

We have all, at one point or another in our lives, bumped into a glass door. If conditions are right; the glass is clean, and there is more transmitted light then reflected light, the glass is almost invisible. The glass is, in fact, not quite invisible. It also reflects light. You can sometime make a glass almost reflectionless, by making the reflections from the back surface interfere destructively with the reflections from the front surface. You can not do that for all wavelengths, but you can do that for some directions. What we want to do here is more ambitious: Make a finite object with $\varepsilon \neq g$ behave as free space no matter from what direction it is viewed.

Lets examine invisibility from the perspective of 8.4, 8.5.

Let us first focus on the equations that involve $\varepsilon$ namely, Eq. 8.5. Consider now free propagation, i.e. no dielectric, in a different curvilinear coordinate system. Let us denote the metric in the second coordinate system $\gamma$ and the corresponding coordinates $\xi$. Eq. 8.5 is now modified by replacing

$$\sqrt{g}\varepsilon^{jk}(x) \mapsto \sqrt{\gamma}\gamma^{jk}(\xi) \tag{8.6}$$

Consider now the case that both $g$ and $\gamma$ describe different coordinates of the same physical space. Then

$$\gamma_{ab} = g_{jk}\frac{\partial x^k}{\partial \xi^a}\frac{\partial x^j}{\partial \xi^b} \tag{8.7}$$

Now, suppose that we choose the the dielectric $\varepsilon$ so that the two *functions* are the same, i.e.

$$\left(\sqrt{g}\varepsilon^{jk}\right)(x) = \left(\sqrt{\gamma}\gamma^{jk}\right)(x) \tag{8.8}$$

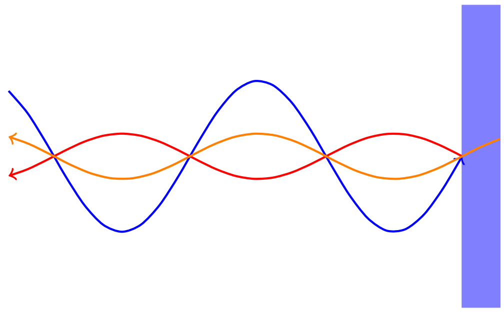


Figure 8.1: Reflection from a mirror can be minimized when the reflection from the front and back interfere destructively. This works for specific directions and wavelengths.

This means that as far as Eq. 8.5 is concerned the dielectric behaves as if its is a coordinate transformation of free space. This is also true for Eq. 8.4 which is independent of $\varepsilon$, since it holds in any curvilinear coordinate system. If $\varepsilon \neq g$ in a finite ball, the coordinate transformation is restricted to the ball and one gets a picture such as in Fig. 8.3. We have thus created an invisible dielectric.

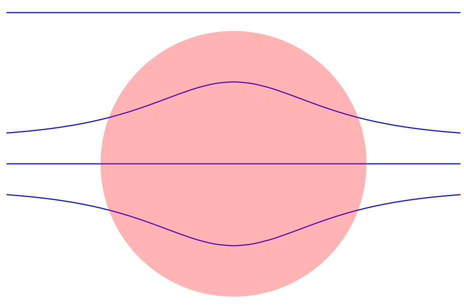


Figure 8.2: The figure illustrates a local coordinate transformation of the Euclidean plane, so that the image of straight lines become the curves in the figure.Such a coordinate transformation can be implemented by a choice of a suitable dielectric tensor $\varepsilon$. The corresponding dielectric, the reddish disk, is invisible.

**Example 8.1.** *Under the local coordinate transformation of polar coordinates*

$$\rho = r + e^{-r} - 1, \quad \theta \mapsto \theta \tag{8.9}$$

*The straight line in the* $\rho, \theta)$ *plane* $\rho \sin\theta = const$ *transforms to* $(r + e^{-r} - 1)\sin\theta = const$ *transforms to a curve with straight asymptotics.*

## 8.3 Cloaking

Now that we know how to make dielectrics that are invisible[3] the next challenge is to engineer dielectrics that can cloak arbitrary objects.

The basic idea behind cloaking is a singular coordinate transformation, that maps the exterior of a ball in physical space to the Euclidean space (with the origin removed).

This is best illustrated by working out through an example in the plane. Let $r$ be the radial coordinate of physical space, which hosts a dielectric tensor $\varepsilon(r)$. Consider the mapping from the plane with radial coordinates $(\rho, \theta)$ into the plane with radial coordinates $(r, \theta)$

$$r^2 = \rho^2 + 1 \tag{8.10}$$

This maps the entire plane $\rho \geq 0$ to the r-plane minus a disk, namely $r \geq 1$.

Physical space $(r, \theta)$ is the Euclidean plane with the standard Eclidean polar metric a $g$. The induced metric $\gamma$ on the $(\rho, \theta)$ plane is, by Eq. 8.7

$$\gamma_{\rho\rho} = \left(\frac{\partial r}{\partial \rho}\right)^2 = \left(\frac{\rho}{r}\right)^2, \quad \gamma_{\theta\theta} = r^2, \quad \det \gamma = \rho^2 \tag{8.11}$$

Maxwell's equation 8.5 is fully determined by the functions

$$\sqrt{\gamma}\gamma^{\rho\rho} = \rho \left(\frac{r}{\rho}\right)^2 = \frac{r^2}{\rho} = \rho + \frac{1}{\rho}, \quad \sqrt{\gamma}\gamma^{\theta\theta} = \rho \left(\frac{1}{r}\right)^2 = \frac{\rho}{\rho^2 + 1} \tag{8.12}$$

The corresponding equation in the physical space with a dielectric is

$$\sqrt{g}\varepsilon^{rr} = r\varepsilon^{rr} \quad \sqrt{g}\varepsilon^{\theta\theta} = r\varepsilon^{\theta\theta} \tag{8.13}$$

The differential equations in the two spaces are the same equations up to renaming $\rho \leftrightarrow r$ if the functions are the same, i.e.

$$r\varepsilon^{rr} = r + \frac{1}{r}, \quad r\varepsilon^{\theta\theta} = \frac{r}{r^2 + 1}, \quad r > 1 \tag{8.14}$$

This is the equation that makes $\varepsilon$ a cloaking material. After simplification

$$\varepsilon^{rr} = 1 + \frac{1}{r^2} = g^{rr} + \frac{1}{r^2}, \quad \varepsilon^{\theta\theta} = \frac{1}{r^2 + 1} = g^{\theta\theta} - \frac{1}{r^2(r^2 + 1)}, \quad \text{for } r > 1 \tag{8.15}$$

The second term makes it manifest that for $r \gg 1$ the dielectric is surrounded by the vacuum. The dielectric $\varepsilon$ for $r < 1$ can be arbitrary and it does not affect the solutions of Maxwell's equations outside the ball $r > 1$. This shows that the enveloping dielectric for $r > 1$ is cloaking the inside.

Of course, a difficult issue we have not addressed is how to engineer $\varepsilon$. In fact, you'd expect a cloaking dielectric to be a wiered material. Landau and Lifshits argue that ordinary materials at thermal equilibrium have

$$\varepsilon^{jk} \geq g^{jk} \tag{8.16}$$

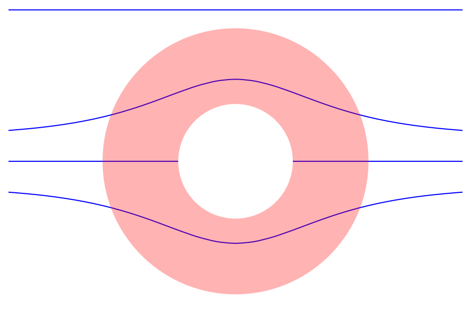


Figure 8.3: The figure illustrates schematically a cloaking envelope shown as a reddish annulus. Anything inside the while disk is invisible from the utside.

This is violated by Eq. 8.15.

**Bibliography** :

J. B. Pendry et. al ."Controlling electromagnetic fields", Science 312, (2006)

[3]The geometric description of cloaking is due to Ulf Leonhardt (Now at Weizmann).

# Chapter 9

# The Stress-Energy tensor

The energy, momentum and angular momentum of the electromagnetic field are identified. Maxwell stress tensor is derived from variation of action due to variations of the metric.

## 9.1 Maxwell stress energy tensor

It is a common experience that the electromagnetic field carries energy: You feel the warmth of the sun. It is not a common experience that the electromagnetic field also carries momentum: You do not feel the sum pushing you. Anyway, you may remember that the energy density, and energy flux (and momentum density) of the electromagnetic field are

$$\mathcal{E} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}, \quad \mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi}, \tag{9.1}$$

We want to derive the covariant generalization of this result.

We first want to argue that what we are looking for is a second rank tensor. This is because in relativity energy and momentum are amalgamated into a

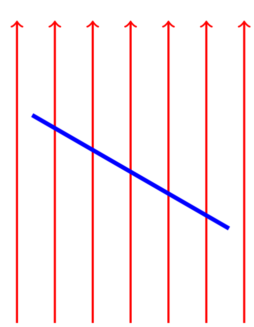


Figure 9.1: The stress tensor represents the flow of momentum (red arrow) through a cross section in space-time (blue line)

single tensor. Therefore energy density and momentum density will be members of the same tensor, and so would be the density of energy currents and density of momentum currents. $T^{00}$ wil stand for the energy density and $T^{0j}$ for the energy current in the j-th direction. $T^{j0}$ stand for the j-th component of the momentum density and $T^{jk}$ for the j-th component of the momentum current in the k-th direction.

The second observation we make is that $T^{\mu\nu}$ must be quadratic in the field $F$. This is because the energy density and Poynting vector have this form.

There are three second rank tensors that we can make that have these properties, using $F$, $\eta$ and $\varepsilon$. $T$ should therefore be a linear combination of

$$F^{\mu\alpha}F^{\nu}{}_{\alpha}, \quad \eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}, \quad \varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu} \tag{9.2}$$

The right linear combination is

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \tfrac{1}{4}\eta^{\mu\nu}F\cdot F\right) \tag{9.3}$$

This is the Maxwell energy-stress tensor. You can easily verify that this is the right choice by checking that

$$T^{00} = \frac{1}{4\pi}\Big(\underbrace{F^{0j}F^{0j}}_{\mathbf{E^2}} - \tfrac{1}{4}\underbrace{F^{\alpha\beta}F_{\alpha\beta}}_{-2(\mathbf{E}^2-\mathbf{B}^2)}\Big) = \mathcal{E} \tag{9.4}$$

Explicitly, $T$ is

$$4\pi T = \frac{\mathbf{E}^2+\mathbf{B}^2}{2}\,\mathbb{1}\underbrace{-}_{relative\ sign}\begin{pmatrix} 0 & \mathbf{B}\times\mathbf{E} \\ \mathbf{B}\times\mathbf{E} & E_iE_j + B_iB_j \end{pmatrix}$$

You recognize the energy density and Poynting vector. We shall discuss the other terms below.

### 9.1.1 The stress-energy and conservation laws

Antennas are pumps of electromagnetic energy and momentum: Antennas allows to convert the motion of charges into electromagnetic field, and and vice versa. The energy and momentum of the field can be converted to the energy and momentum of the particle. For a charged particle we found for the 4-force vector, Eq. 4.17

$$f_\nu = \frac{e}{c}F_{\nu\mu}u^\mu, \quad p_\mu = mu_\mu = m\gamma(-c,\mathbf{v}) \tag{9.5}$$

By Newton's equation

$$\dot{p}_\nu = f_\nu \tag{9.6}$$

so the 4-force measures the rate of gain of the 4-momentum of the particles. The rate of change of the energy is $cf_0$.

The loss of $\nu$-momentum from the field is measured by its divergence

$$\partial_\mu T^{\mu\nu} = \partial_\mu T^{\nu\mu} \tag{9.7}$$

since $T$ is symmetric. The loss of momentum of the field is the gain of momentum of the particle, in the case of continuous charge distribution, we must have,

$$\partial_\mu T^{\mu\nu} = \frac{1}{c} j_\mu F^{\mu\nu} \tag{9.8}$$

$T$ has dimension of energy density (in space) and $\partial T$ has dimension of energy density (in space-time). This gives the rhs of Eq. 9.8 the interpretation of source of energy per unit of space-time volume. The identity follows from Maxwell's equations. Since this is a subtle and computation let us first observe that since $T$ is bi-linear in $F$ and Maxwell's equations are linear with $j$ its source term the rhs has the structure one expects.

As preparation let us fist compute the divergence of the first term in $T$

$$\begin{aligned}
\partial_\mu (F^{\mu\alpha} F^\nu{}_\alpha) &= (\partial_\mu F^{\mu\alpha}) F^\nu{}_\alpha + F^{\mu\alpha} \partial_\mu F^\nu{}_\alpha \\
&= -\frac{4\pi}{c} j^\alpha F^\nu{}_\alpha + F^{\mu\alpha} \partial_\mu F^\nu{}_\alpha \\
&= \frac{4\pi}{c} j_\mu F^{\mu\nu} + F_{\mu\alpha} \partial^\mu F^{\nu\alpha} \\
&= \frac{4\pi}{c} j_\mu F^{\mu\nu} + F_{\alpha\beta} \partial^\alpha F^{\nu\beta}
\end{aligned}$$

We have used the the in-homogeneous Maxwell Eq. 6.8 in the second line, moved indexes up and down in the third and renamed $\mu \mapsto \alpha$ and $\alpha \mapsto \beta$ in the last line. For the second term we have

$$\begin{aligned}
\partial_\mu (\eta^{\mu\nu} F \cdot F) &= 2 F_{\alpha\beta} \partial^\nu F^{\alpha\beta} \\
&= -2 F_{\alpha\beta} (\partial^\alpha F^{\beta\nu} + \partial^\beta F^{\nu\alpha}) \\
&= -2 F_{\alpha\beta} \partial^\alpha F^{\beta\nu} - 2 F_{\beta\alpha} \partial^\alpha F^{\nu\beta} \\
&= 4 F_{\alpha\beta} \partial^\alpha F^{\nu\beta}
\end{aligned}$$

where I have used the homogeneous Maxwell equation in the second line in the form of Exercise 3.15. In the third line I replaced $\alpha \leftrightarrow \beta$. Putting these in the equation for $T^{\mu\nu}$ we get Eq. 9.8.

**Exercise 9.1** (Plane waves)**.** *Show that the stress tensor for plane electromagnetic waves*

$$A_\mu = a_\mu e^{ik\cdot x}, \quad k^\mu A_\mu = 0, \quad k_\mu k^\mu = 0$$

*is*

$$4\pi T^{\mu\nu} = a \cdot a \; k^\mu k^\nu$$

**Exercise 9.2.** *Compute* $\partial_\mu (\mathbf{E} \cdot \mathbf{B})$

## 9.2 Conservation laws

In the absence of currents $j$, the energy-momentum of the field is conserved. This follows from

$$\partial_\mu T^{\mu\nu} = 0 \tag{9.9}$$

To see this consider a space-time box as in Fig. 9.2. Suppose the spatial box

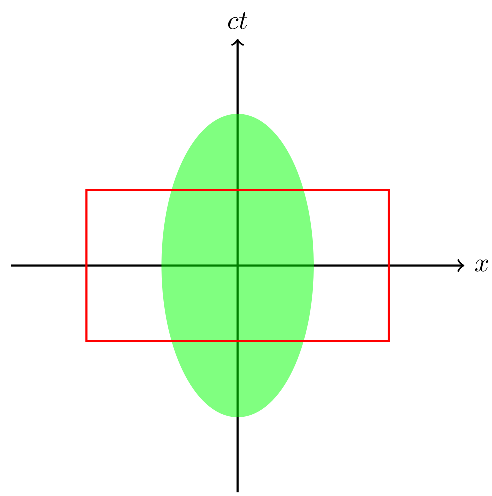



Figure 9.2: The field is represented by the green ellipse and the space-time box by the red rectangle

is large enough to embrace all the fields at any given time. Then

$$0 = \int d\Omega\, \partial_\mu T^{\mu\nu} = \int dS_\mu T^{\mu\nu} = \int dS_0 T^{0\nu}\Big|_{t_1}^{t_2} = \int dV\, T^{0\nu}\Big|_{t_1}^{t_2}$$

It follows that the 4 vector

$$\int dV\, T^{0\nu}$$

is conserved in time. We identify $T^{00}$ with the energy density and $T^{0j}/c$ with the momentum density:

$$T^{00} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}, \quad T^{0j} = \frac{\mathbf{B}\times\mathbf{E}}{4\pi} \tag{9.10}$$

The Poynting vector is the momentum density up to factor $c$.

It may be worthwhile to note that even though $T$ is symmetric the physical interpretation of $T^{\mu\nu}$ is different from the interpretation of $T^{\nu\mu}$. For example $T^{0j}$ is interpreted as the momentum density (up to factor $c$) while $T^{j0}$ is interpreted as the energy flux.

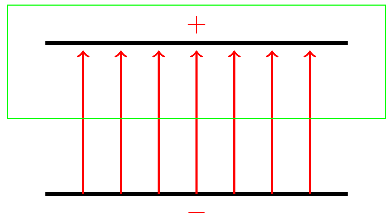



Figure 9.3: You need to apply an external force to hold the capacitor plates from collapsing on each other.

## 9.3 Stress tensor

The spatial part of the stress-tensor is

$$8\pi T = \begin{pmatrix} -E_x^2 + E_y^2 + E_z^2 & -2E_xE_y & -2E_xE_z \\ -2E_yE_x & E_x^2 - E_y^2 + E_z^2 & -2E_yE_z \\ -2E_zE_x & -2E_zE_y & E_x^2 + E_y^2 - E_z^2 \end{pmatrix} \tag{9.11}$$

The term $T^{k3}$ gives the density of the k-th momentum in the space-time volume element

$$dP^k = T^{k3} dx^1 dx^2 dt \tag{9.12}$$

Since the force is the rate of momentum the k-th component of the force $dF^k$ acting on the 3-surface $dx^1 dx^2$ is

$$dF^k = T^{k3} dx^1 dx^2$$

This gives $T^{jk}$ the same meaning as the stress tensor in the theory of elasticity.

### 9.3.1 Case study: Capacitor plates

The two sides of Eq. 9.8 give us two ways to compute the forces in an electromagnetic fields. Let us illustrate this with the force of attraction between the two capacitor plates in the figure.

*Using the rhs of Eq. 9.8*: The rhs gives the energy density per space time volume. Consider the volume associated with the green box. The 4-current has only the 0-component, given by the surface charge density, i.e.

$$j_0 = c\sigma\delta(z - z_0), \quad 4\pi\sigma = -E_z \tag{9.13}$$

and we took the top plate to be at $z_0$. The field is discontinuous at $z_0$

$$F^{0z} = \begin{cases} 0 & z > z_0 \\ E_z & z < z_0 \end{cases} \tag{9.14}$$

The product of a delta function with a discontinuous function is mathematically problematic but the natural physical choice is to take the mean

$$\frac{1}{c}\int dz j_0 F^{03} = \tfrac{1}{2}\sigma E_z = -\frac{E_z^2}{8\pi} \tag{9.15}$$

The force per unit area pulls the upper plate down, indicated by the minus sign,

$$-\frac{E_z^2}{8\pi} \tag{9.16}$$

*Using the lhs of Eq. 9.8*: The stress $T^{zz}$ is

$$T^{zz} = \begin{cases} -\frac{E_z^2}{8\pi} & \text{bottom plate} \\ 0 & \text{top plate} \end{cases} \tag{9.17}$$

The bottom of the green box has space-time volume

$$dS_3 = -dx^1 dx^2 cdt \tag{9.18}$$

The rate of flow of z-momentum out of the green box per unit area is

$$\frac{E_z^2}{8\pi} \tag{9.19}$$

Since the box is loosing z-momentum, the force in the z-direction is negative. As the green box can be shrunk to hug the top plate, the force on the plate computed in both ways agree.

## 9.4 Field lines as rubber bands

Let us look at

$$8\pi T^{xx} = E_y^2 + E_z^3 - E_x^2 + B_y^2 + B_z^3 - B_x^2$$

The parallel and perpendicular components come with opposite signs: The stress can be positive or negative, but the sign has nothing to do with the signs of **E**. When the stress is negative the electromagnetic field provides negative pressure.

**Exercise 9.3.** *Compute the components of $T$ for the example 6.2.Show that the current carrying wire pumps energy into the electromagnetic field at a constant rate. Where does the energy come from and where is it dumped?*

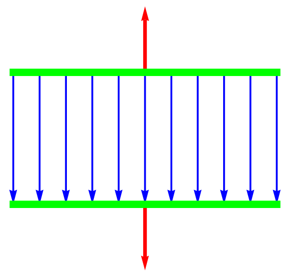


Figure 9.4: You need to apply a force to hold two oppositely charged capacitor plates apart (red arrows). If the arrow is in the z-direction, $T^{zz} < 0$. You get this sign if you think of the electric field lines as rubber band: As if the pressure is negative.

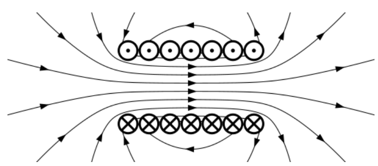


Figure 9.5: The magnetic field lines of a solenoid run parallel to the solenoid axis. As a consequence the stress in the radial direction $T^{\rho\rho} > 0$ inside the solenoid. You get the right sign if you replaced the field lines by rubber bands: Stretched rubber bands along the z-axis, that fan out in the radial direction, will lead to a positive pressure in the radial direction and negative pressure in the axial direction.

## 9.5 The stress tensor as variation of the metric

When Maxwell constructed his theory the queen of science was, of course, mechanics. In particular, he understood well elasticity and fluid mechanics. In elasticity theory the concepts of stress and strain are important, and it was natural for Maxwell to ask what is their analogs in electrodynamics. One can think of a strain as a deformation of the metric. For example, the strain shown in the figure [1] can be represented by deformation of the Euclidean metric

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to g = \begin{pmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon \end{pmatrix}$$

[1]The vector field is divergence-less and curl-free.

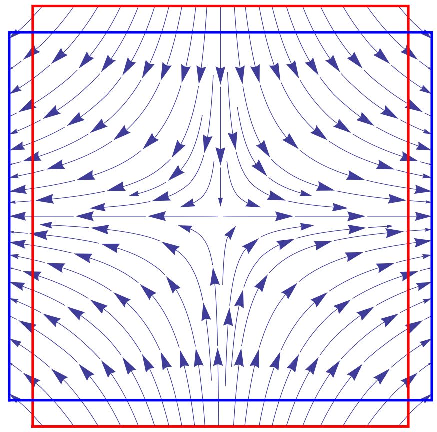


Figure 9.6: In the theory of elasticity a strain is described by a vector field. The figure shows the vector field associated with (uniform) contraction of $y$ and dilation of $x$: Namely $(x, -y)$. The strain causes stress in the material. Energy is stored in it like in a compressed spring.

### 9.5.1 Variation of the metric in mechanics

The Lagrangian of a free classical particle is

$$L = \frac{m}{2} g_{ij} \dot{q}^i \dot{q}^j, \quad p_j = \frac{\partial L}{\partial \dot{q}^j} = m g_{jk} \dot{q}^k = m \dot{q}_j$$

The variation of the metric gives

$$\frac{\partial L}{\partial g_{ij}} = \tfrac{1}{2}\, m \dot{q}^i \dot{q}^j = \tfrac{1}{2} p^i \dot{q}^j$$

The symmetric tensor describes the i-momentum current in the j direction.

### 9.5.2 Variation of the metric in electrodynamics

The action depends on the metric in two places. First, in the volume element

$$d\Omega = \sqrt{|g|} dx^0 dx^1 dx^2 dx^3$$

and also in the scalar product

$$F \cdot F = F^{\alpha\beta} g_{\beta\gamma} F^{\gamma\delta} g_{\delta\alpha} \tag{9.20}$$

Hence

$$\begin{aligned}
\delta\big(\sqrt{|g|} F \cdot F\big) &= \delta(\sqrt{|g|})\ F \cdot F + \sqrt{|g|} F^{\alpha\beta} F^{\gamma\delta} \delta(g_{\beta\gamma} g_{\delta\alpha}) \\
&= \delta(\sqrt{|g|})\ F \cdot F + 2\sqrt{|g|} F^{\alpha\beta} F^{\gamma\delta} g_{\delta\alpha} \delta g_{\beta\gamma} \\
&= \delta(\sqrt{|g|})\ F \cdot F + 2\sqrt{|g|} F^{\alpha\beta} F^{\gamma}{}_{\alpha} \delta g_{\beta\gamma}
\end{aligned}$$

### 9.5.3 Matrix calculus

To compute the variation of $\det g$ we need tools from matrix calculus. Being symmetric $g$ can be diagonalized

$$g = \sum P_\mu \gamma_\mu, \quad P_\mu = |\mu\rangle\langle\mu|$$

where $\gamma_\mu$ are its (real) eigenvalues and $P_\mu$ are orthogonal projections. By definition

$$\det g = \prod \gamma_\mu \Longrightarrow \log|g| = \sum \log \gamma_\mu$$

and so

$$\delta \log \sqrt{\det g} = \tfrac{1}{2}\delta \log(|g|) = \tfrac{1}{2}\sum \frac{\delta\gamma_\mu}{\gamma_\mu}$$

We want to express the right hand side in terms of $g$ and its variation $\delta g$. To do that observe that,

$$g^{-1} = \sum \frac{P_\mu}{\gamma_\mu}$$

and so

$$g^{-1}\delta g = \sum \frac{P_\mu\left(P_\nu \delta\gamma_\nu + \gamma_\nu \delta P_\nu\right)}{\gamma_\mu}$$

**Exercise 9.4** (Projections)**.** *Show that if $P_\mu$ are orthogonal projections, $P_\mu P_\nu = \delta_{\mu\nu}P_\nu$, then*

$$Tr(P_\mu \delta P_\nu) = 0$$

We have then showed that

$$\delta \log \sqrt{-\det g} = \tfrac{1}{2}Tr(g^{-1}\delta g) = \tfrac{1}{2}g^{\gamma\beta}\delta g_{\beta\gamma}$$

and so finally

$$\delta\sqrt{|g|} = \sqrt{|g|}\,\delta \log \sqrt{|g|} = \tfrac{1}{2}\sqrt{|g|}g^{\gamma\beta}\delta g_{\beta\gamma} \tag{9.21}$$

### 9.5.4 The stress tensor

Collecting terms we get

$$\frac{\partial(\sqrt{|g|}F\cdot F)}{\partial g_{\beta\gamma}} \quad = \quad \sqrt{|g|}\Big(F^{\beta\alpha}F^{\gamma}{}_{\alpha} - \tfrac{1}{4}g^{\beta\gamma}F^{\mu\nu}F_{\mu\nu}\Big) \tag{9.22}$$

We now shift back $\sqrt{g}$ into the volume element and recover the energy momentum tensor.

## 9.6 Nöther: Symmetries

Conservation laws expresses symmetries, a relation due to Nöther. If you compute the action of a given field configuration in a box, you get the same number if you translate (and rotate) the coordinates. This is a trivial statement, following from the homogeneity of Minkowski space-time. You may expect to get no any interesting identities from it. The genius of Nöther was to realize that if the field is not an arbitrary field configuration, but one that solves the Euler Lagrange equations, then the statement reduces to statement about the fields on the boundaries of the box. This is what you expect form a conservation law: What comes in through one boundary must leave through another.

Nöther introduced a technique which is amusing: She split the real operation of coordinate shift into two virtual operations: One that shifts the box–but not the fields–and one that shifts the fields–but not the box. Think of moving your bag in two steps: First you move the content of the bag and second you move the empty bag. This relates to non-trivial quantities. For fields that satisfy Euler-Lagrange equations, the variation live on the boundary.

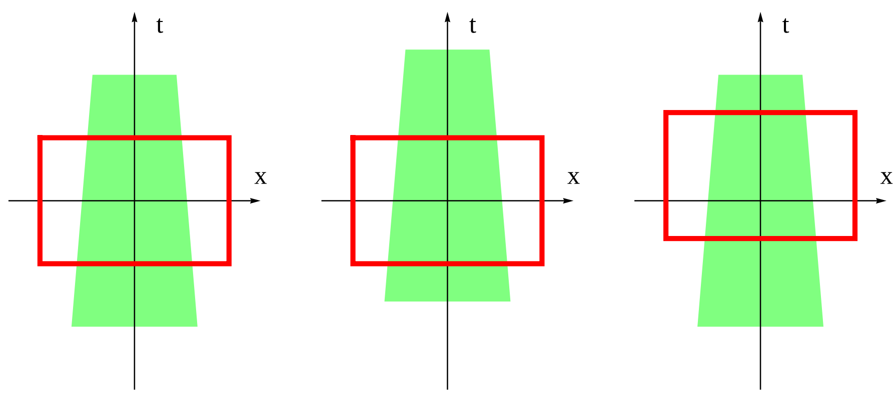



Figure 9.7: The action remains the same when the fields and the integration box are both shifted in space-time. For a small shift the change in action can be split into two virtual shifts: A shift of the field with the box held fixed, shown in the middle figure, and a shift of the box with the field held fixed field, shown on the right.

### 9.6.1 Shifting the field

Consider the change in action due to a shift of the field without a shift of the box. The variation of the action $S_F$ due to arbitrary variation $\delta A_\mu$ of the fields in a *a fixed box*, has been computed in Eq. 6.5 and was found to be

$$4\pi c\ (\delta S_F) = -\underbrace{\int d\Omega\, \partial_\mu (F^{\mu\nu}\delta A_\nu)}_{\text{bdry term}} + \underbrace{\int d\Omega (\partial_\mu F^{\mu\nu})\delta A_\nu}_{\text{Euler-Lagrange}} \tag{9.23}$$

For fields that satisfy the Euler-Lagrange equation, only the left term matter–the variation is a boundary term

$$4\pi c\ (\delta S_F) = -\int d\Omega\, \partial_\mu (F^{\mu\nu}\delta A_\nu) \tag{9.24}$$

A uniform space-time shift by $\delta\xi^\alpha$ leads to variation in the fields

$$\delta A_\mu(x) = -(\partial_\alpha A_\mu)\delta\xi^\alpha \tag{9.25}$$

**Exercise 9.5** (Signs-Sigh)**.** *Explain the minus sign.*

Inserting the variation into Eq. (9.24) and using Maxwell equation[2] one finds for the integrand

$$\begin{aligned} -\partial_\mu (F^{\mu\nu}\delta A_\nu) &= \partial_\mu (F^{\mu\nu}\partial_\alpha A_\nu)\delta\xi^\alpha \\ &= \partial_\mu \big(F^{\mu\nu}F_{\alpha\nu}\big)\delta\xi^\alpha + \partial_\mu \big(F^{\mu\nu}\partial_\nu A_\alpha\big)\delta\xi^\alpha \\ &= \partial_\mu \big(F^{\mu\nu}F_{\alpha\nu}\big)\delta\xi^\alpha + \big(F^{\mu\nu}\partial_{\mu\nu} A_\alpha\big)\delta\xi^\alpha \\ &= \partial_\mu \big(F^{\mu\nu}F_{\alpha\nu}\big)\delta\xi^\alpha \end{aligned}$$

We see that the change in action due to shifting the field is:

$$4\pi c\ (\delta S_F) = \int d\Omega\, \partial_\mu \big(F^{\mu\nu}F_{\alpha\nu}\big)\delta\xi^\alpha \tag{9.26}$$

### 9.6.2 Shifting the box

As a warmup consider the variation of a one dimensional integral upon shifting boundary points by $\delta\xi$

$$\delta\left(\int_a^b f(x)dx\right) = \int_{a+\delta\xi}^{b+\delta\xi} f(x)dx - \int_a^b f(x)dx = \delta\xi\big(f(b) - f(a)\big)$$

The case at hand is the multidimensional version of this.

Shifting the box without shifting the fields changes the action by boundary terms of the form

$$\begin{aligned} 4\pi c\ (\delta S_F) &= -\tfrac{1}{4}\left(\int dS_\alpha\, F\cdot F\right)\delta\xi^\alpha \\ &= -\tfrac{1}{4}\int d\Omega\, \partial_\alpha\,(F\cdot F)\,\delta\xi^\alpha \end{aligned} \tag{9.27}$$

and we used the fundamental theorem of calculus in the second line.

[2] Note that $\xi^\alpha$ is a constant, not a function.

### 9.6.3 Joint box and field shift

For the joint shift we get by combining Eq. 9.27 and 9.26

$$
\begin{aligned}
0 &= \int d\Omega \left(\partial_\mu \left(F^{\mu\nu} F_{\alpha\nu}\right) - \tfrac{1}{4}\partial_\alpha \left(F \cdot F\right)\right)\delta\xi^\alpha \\
&= \int d\Omega \, \partial_\mu \left(F^{\mu\nu} F_{\alpha\nu} - \tfrac{1}{4} g^\mu_\alpha \left(F \cdot F\right)\right)\delta\xi^\alpha \\
&= \int d\Omega \left(\partial_\mu T^{\mu\alpha}\right)\delta\xi_\alpha
\end{aligned}
$$

Since this is supposed to hold for any (infinitesimal) box and any shift, we get the conservation law (in the absence off source currents).

### 9.6.4 Symmetry and traceless

The tensor is symmetric and traceless. Symmetry is obvious from the definition, as $\eta$ is symmetric. Traceless follows from the fact that ${\eta^\alpha}_\alpha = 4$ and so

$$
{T^\alpha}_\alpha = \frac{1}{4\pi}\left(F^{\alpha\mu} F_{\alpha\mu} - \tfrac{1}{4}{\eta^\alpha}_\alpha F^{\mu\nu} F_{\mu\nu}\right) = 0
$$

## 9.7 Applications

### 9.7.1 Radiation pressure

The luminosity of the sun $L_\odot = 3.6 \times 10^{26}\ [W]$, giving a stream of $10^{45}$ photons/sec. The radial component of Maxwell energy momentum tensor at a distance $R$ from the sun is then

$$
T_{0\hat{r}} = \frac{L_\odot}{4\pi R^2 c} \tag{9.28}
$$

Consider a macroscopic (black) particle of radius $r$ that perfectly absorbs radiation. The force on the particle at a distance $R$ from the sun is then

$$
F_{radiation} = \frac{r^2}{4R^2 c} L_\odot \tag{9.29}
$$

The gravitational force on a particle with density $\rho$ is

$$
F_{gravity} = \frac{4\pi\rho r^3 M_\odot G}{3R^2} \tag{9.30}
$$

Where $M_\odot = 2 \times 10^{33}\ [gram]$ and Newton constant $G = 6.7 \times 10^{-8}\ [cgs]$. The ratio of the two is then

$$
\frac{F_{radiation}}{F_{gravity}} = \frac{3L_\odot}{16\pi r\rho c M_\odot G} \approx \frac{0.06\,[mgr/cm^2]}{\rho r} \tag{9.31}
$$

For water $\rho = 1$ $[gm/cm^3]$. For earth, $r = 6 \times 10^8$ $[cm]$, the ratio is minuscule: $10^{-13}$. However, for very small grains, of radius less than $6 \times 10^{-5}$ $[cm]$ radiation dominates.

Radiation pressure cleans the solar neighborhood from fine dust. This could be a mechanism of transporting viruses from our solar system to distant parts of the universe[3].

**Exercise 9.6** (Comet tails)**.** *Can you figure out the shape of a comet tail? Suppose the tail is associated with a planet in circular non-relativistic orbit.* Hint: Figure out the tail in the rotation frame.

### 9.7.2 Solar sails

The computation above can be applied to solar sails. A sail of area $A$ and width $d$ can be used to sail away from the sun provided

$$F_{radiation} = \frac{A}{4\pi R^2 c} L_\odot > F_{gravity} = \frac{\rho A d M_\odot G}{R^2} \tag{9.32}$$

Cancelling the similar terms we get

$$\frac{1}{4\pi c} L_\odot > \rho d M_\odot G \tag{9.33}$$

Up to factors of order unity we get, the same estimate as above. You need very thin sails to build solar sails.

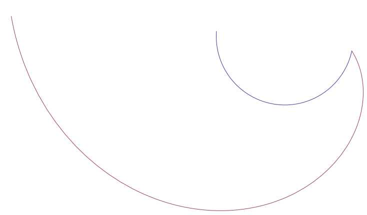


Figure 9.8: A planet encircling a star and the tail of dust it sprays (tail)

### 9.7.3 Halbach array

The Halbach array is shown in the fig 9.9. Make a qualitative plot of its field lines. You will find a proper plot here. Since the field is large and essentially parallel to the array on one side of the array, and small on the other. You may then wonder if the array is a Baron von Munchhausen: Properly oriented, it will float in gravitational field. Discuss the Maxwell stress tensor and resolve this apparent paradox.

[3]The assumption that the particle is black is not reasonable when the radiation penetrates a distance comparable to the size.

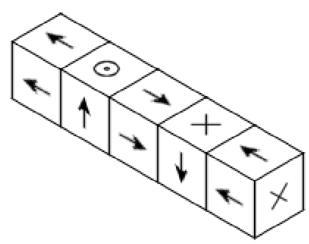


Figure 9.9: Halbach array gives a large magnetic field above the array and small one below it.

# Chapter 10

# Electrostatics and magnetostatics

Now, that we have Maxwell equations, it remains to look at solutions for interesting physical problems. We start with time independent problems.

## 10.1 Static electric fields:

The electric field $\mathbf{E}$ is determined by Gauss and Faraday's law

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{E} + \underbrace{\dot{\mathbf{B}}}_{=0} = 0$$

and dot is a derivative with respect to $x^0 = ct$. In the static case $\dot{\mathbf{B}} = 0$ and the equations reduce to

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{E} = 0$$

The source $\rho(\mathbf{x})$ is assumed to be known.

These two equations[1] determine $\mathbf{E}$ completely. Faraday's law implies that

$$\mathbf{E} = -\nabla\phi$$

Substitution in Gauss law gives Poisson's equation

$$\Delta\phi = -4\pi\rho \tag{10.1}$$

which is a partial differential equation for the potential (if the dimension $n \geq 2$).

**Remark 10.1** (Time dependent $\rho$)**.** *From a formal mathematical point of view, one may also consider Poisson's equation with $\rho$ that is time dependent (e.g. a moving point charge). However, time dependent $\rho$ would normally entail $\dot{\mathbf{B}} \neq 0$.*

[1] And the assumption that space is Minkowski and the fields vanish at infinity

## 10.2 Harmonic functions

Let us start with the simple, but important, case where $\rho = 0$. Functions that satisfy

$$\Delta\phi = 0 \tag{10.2}$$

are called Harmonic. They are dear to both physicists and Mathematicians.

A fundamental fact about Harmonic functions is:

**Theorem 10.2** (Harmonic functions)**.** *If $\phi$ is Harmonic, then $\phi(\mathbf{x})$ is the average value of $\phi$ on a sphere centered at $\mathbf{x}$.*

The theorem is evident in one dimension: A harmonic function is a linear function and hence the average of equidistant neighbors. We shall postpone the proof in the general case after we assemble some more tools. Let us first consider an important consequence:

**Corollary 10.3** (Erenshaw)**.** *Harmonic functions in a domain $\omega$ assume (local) maxima and minima on the boundary $\partial\Omega$. As a consequence, charges can not be stably trapped by electrostatic fields.*

**Exercise 10.4.** *Suppose that locally*

$$E_j(x) = e_j^{(0)}(0) + e_{jk}^{(2)} x^k + O(x^2)$$

*Show that $\nabla \cdot \mathbf{E} = 0$ implies $Tr\ e^{(2)} = 0$. In particular the matrix $e^{(2)}$ can not have all eigenvalues of one sign.*

### 10.2.1 Beating Ehrenshaw: Magnetic and electric traps

Levitron is a spinning magnetic dipole that hovers in a gravitational field above a stationary magnet. You may wonder if this toy violate Ehrenshaw. It does not, because the top is spinning and Ehrenshaw deals with a stationary case. But, this does not yet explain how Levitron works. Let's look at the problem in more detail.

The energy of a stationary magnetic dipole $\mathbf{d}$ whose mass is $M$ in electrostatic fields $\mathbf{E}$ and gravitational field $\mathbf{g}$ is

$$\mathcal{E} = -\mathbf{d} \cdot \mathbf{E} + M\mathbf{g} \cdot \mathbf{x}$$

The total force acting on the dipole is

$$\mathbf{F} = -\nabla\mathcal{E} = \nabla(\mathbf{d} \cdot \nabla\phi) - M\mathbf{g} \tag{10.3}$$

The dipole is in equilibrium if $\mathbf{F} = 0$. Suppose we found a point where $\mathbf{F} = 0$. We claim that such a point of equilibrium is unstable if $\mathbf{d}$ is a fixed vector and $\phi$ is harmonic, which it must be in the electrostatic case. Indeed:

$$\Delta\mathcal{E} = \nabla \cdot \mathbf{F} = \Delta(\mathbf{d} \cdot \nabla)\phi = (\mathbf{d} \cdot \nabla)\Delta\phi = 0 \tag{10.4}$$

How can we turn this setting to one where the equilibrium is stable? Clearly, this can only occur if we let $\mathbf{d}$ be a function of position. Suppose that, for some reason, the dipole wants to orient itself opposite to the local field,

$$\mathbf{d}(\mathbf{x}) = -d\,\hat{\mathbf{E}}(\mathbf{x})$$

Eq. 10.4 is modified to

$$\Delta\mathcal{E} = \nabla\cdot\mathbf{F} = d\Delta|\mathbf{E}| \tag{10.5}$$

$|\mathbf{E}|$ is, of course, not a harmonic function, even if $\phi$ is. For example, near a point where $|\mathbf{E}| = 0$, the function looks like a cone, with a minimum at the vertex of the cone. This removes the in-principle bar from finding a stable equilibrium. This, of course, does not yet guarantee stability and it turns out that the analysis is involved, so we stop here.

## 10.3 Laplace equation in two dimensions

Two dimensions are special in that the machinery of holomorphic functions allows to compute things explicitly in many cases.

A point in the plane can be described by a the complex number $z = x + iy$ and $\bar{z} = x - iy$. Here we regard $z$ and $\bar{z}$ as independent variables. Similarly,

$$2\partial = 2\partial_z = \partial_x - i\partial_y, \quad 2\bar{\partial} = 2\partial_{\bar{z}} = \partial_x + i\partial_y, \tag{10.6}$$

This language also works for any real vector in 2-D, which can be represented by a single complex number

$$E = \mathbf{E}\cdot\hat{\mathbf{x}} + i\,\mathbf{E}\cdot\hat{\mathbf{y}}$$

A (complex) function $f(x,y)$ can be written as a function of $f(z,\bar{z})$. Analytic functions are functions of $z$ only and

$$\bar{\partial}f = 0 \tag{10.7}$$

is the Cauchy-Riemmann equation for analytic functions.

In two dimensions, the vector field $\nabla\phi$ for real $\phi$, can be identified with a single complex function. Clearly, it must be some linear combination of $\partial\phi$ and $\bar{\partial}\phi$. From Eq. 10.6, one finds that actually

$$E = -2\bar{\partial}\phi \tag{10.8}$$

Now, in two dimensions both $\nabla\cdot\mathbf{E}$ and $\nabla\times\mathbf{E}$ are (scalar valued) functions which must be a linear combination of $\partial E$ and $\bar{\partial}E$. A computation shows that

$$2\partial E = \nabla\cdot\mathbf{E} - i\nabla\times\mathbf{E} \tag{10.9}$$

Hence, for any real vector field, $\nabla\cdot\mathbf{E}$ is the real part of $2\partial E$.

An easy computation gives for the Laplacian

$$\Delta\phi = 4\partial_z\bar{\partial}_z\phi \tag{10.10}$$

A harmonic function in two dimensions is therefore of the form

$$\Delta\phi = 0 \longrightarrow \phi(z,\bar{z}) = f(z) + g(\bar{z})$$

The general real solution is therefore

$$\phi(z,\bar{z}) = f(z) + f(\bar{z})$$

### 10.3.1 Harmonic functions and Riemann mapping

Harmonic functions in a domain $D$ are determined by their values on the boundary $\partial D$.

Let us first see how this works for the a unit disc. We assume that the boundary values are given by the Fourier series[2]

$$\phi(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

Write the potential inside the disc as a sum of a holomorphic and anti-holomorphic functions

$$\phi = f + \bar{g} + a_0, \quad f(z) = \sum_{n=1}^{\infty} f_n z^n, \quad g(z) = \sum_{n=1}^{\infty} g_n z^n$$

Evidently, for $n \geq 1$

$$f_n = a_n, \quad \bar{g}_n = a_{-n}$$

$f$ and $g$ are analytic in the unit disc since the corresponding series are absolutely convergent.

We can now use Riemann mapping theorem to generalize this result for the unit disc to any open, simply connected domain $D$ in the plane. Riemann mapping theorem states that if $D$ is an open, simply connected domain in $\mathbf{R}^2$ then there is a holomorphic (and invertible) function $\zeta = \varphi(z)$ that maps $D$ to the unit disc $|\zeta| < 1$.

## 10.4 Harmonic polynomials and spherical harmonics

Consider homogeneous polynomials of degree $n \geq 0$ in $d$ variables. They form a vector space $V_{n,d}$ with elements

$$x_1^{n_1} \dots x_d^{n_d}, \quad n = \sum_{j=1}^{d} n_j, \quad n_j \geq 0$$

[2] We therefore assume that $\sum |a_n| < \infty$. Note that $\phi$ need not be a continuous function on the boundary.

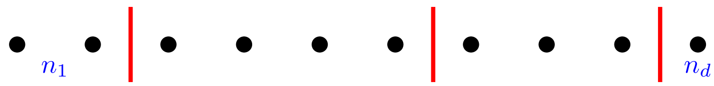



Figure 10.1: The $n$ balls correspond to the degree of the polynomial. $(d-1)$ red partitions define $d$ boxes. The $j$-th box represents $x_j$. The number of dots in the j-th box gives the power of $x_j$. The number of different polynomials is the same as the the number of ways of putting the $d-1$ red partitions among the $n$ balls. There are $n+d-1$ place holders, and we need to decide if we put a partition or a ball in the place-holder. This is the problem of selecting $d$ objects out of $n+d-1$ objects, and so is is the binomial coefficient Eq. 10.11.

The dimension of this space is

$$\dim V_{n,d} = \binom{n+d-1}{d-1} \tag{10.11}$$

This is explained in the figure. The Harmonic polynomial are the kernel of the Laplacian[3]

$$\Delta : V_{n,d} \to V_{n-2,d} \tag{10.12}$$

The homogeneous Harmonic polynomials of degree $d$ are a vector space $H_{n,d}$ and its dimension is evidently

$$\dim H_{n,d} = \dim V_{n,d} - \dim V_{n-2,d} = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1}$$

The harmonic polynomial of degree $n$ can be written as

$$H_n(x_1, \ldots, x_d) = r^n Y_n(\Omega) \tag{10.13}$$

where $\Omega$ denotes a point on the unit sphere in $d$ dimension. $Y$ is a spherical Harmonic. As we shall now see, it is an eigenfunction of the "spherical Laplacian".

The Laplacian in spherical coordinates splits into a radial and angular pieces by

$$\Delta = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} r^{d-1} \frac{\partial}{\partial r} - \frac{1}{r^2} L^2, \quad d \geq 2 \tag{10.14}$$

The second term is the angular part of the Laplacian. $L^2$ may be thought of as (minus) the Laplacian on the unit sphere, or alternatively as the kinetic energy associated with the angular motion.

[3] $V_{n-2,d} = 0$ for $n = 0, 1$ since $\binom{n+d-3}{d-1} = 0$ for $n < 2$.

Since $H_n$ is homogeneous and harmonic

$$\begin{aligned}
0 = \Delta H_n &= \frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}H_n - \frac{1}{r^2}L^2 H_n \\
&= Y_n \frac{1}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1}\frac{\partial}{\partial r}r^n - r^{n-2}L^2 Y_n \\
&= r^{n-2}\Big(n(n+d-2) - L^2\Big)Y_n
\end{aligned} \tag{10.15}$$

It follows that the "spherical harmonics" $Y$ are eigenfunctions of $L^2$ with eigenvalues

$$L^2 Y_n = n(n+d-2)Y_n, \quad d \geq 2 \tag{10.16}$$

There is a single harmonic function on the sphere, namely, the constant function corresponding to eigenvalue $n = 0$ of the "spherical Laplacian". For $n \geq 1$ the dimension of the eigenspace of $L^2$ is larger than 1 and there is freedom in choosing a basis which leads to different conventions in the definitions of spherical harmonics.

**Exercise 10.5.** *In two dimensions the space of Harmonic polynomial of degree $n$ is generated by*

$$z^n = (x+iy)^n, \quad \bar{z}^n = (x-iy)^n,$$

*The "spherical Laplacian" is*

$$L^2 = -\frac{\partial^2}{(\partial\theta)^2}$$

*and the "spherical harmonics" are*

$$e^{\pm in\theta}, \quad \dim H_{n,2} = 2 \quad n \geq 1$$

**Exercise 10.6.** *In three dimensions*

$$\dim H_{n,3} = 2n+1$$

*and the spectrum of "spherical Laplacian" is*

$$Spectrum(L^2) = n(n-1)$$

*In three dimensions we can write the Laplacian as*

$$\Delta = \frac{d^2}{dz^2} + 4\partial\bar{\partial} \tag{10.17}$$

*It follows that $(x+iy)^n$ and $(x-iy)^n$ are Harmonic. In particular*

$$z^n = r^n\left(\frac{x+iy}{r}\right)^n \Longrightarrow Y_{n,n}(\theta,\phi) = e^{i\phi}\sin^n\theta \tag{10.18}$$

*up to normalization and similarly for $\bar{z}$.*

### 10.4.1 Harmonic functions and multipoles

Inversion is the map

$$\mathbf{x} \mapsto \frac{\mathbf{x}}{r^2}, \quad r^2 = \mathbf{x} \cdot \mathbf{x} \tag{10.19}$$

**Exercise 10.7.** *Show that under inversion a sphere in $\mathbb{R}^n$ is mapped to a sphere.*

Here is an interesting fact I have learned from Barak Katzir

**Theorem 10.8.** *Suppose $\phi(\mathbf{x})$ is a Harmonic function in $\mathbb{R}^d$ then*

$$\psi(\mathbf{x}) = r^{2-d}\phi\left(\frac{\mathbf{x}}{r^2}\right), \quad r^2 = \mathbf{x} \cdot \mathbf{x} \tag{10.20}$$

*is Harmonic.*

Clearly, it is enough to prove this for a dense set of Harmonic functions, and so we focus on proving instead that for a homogeneous harmonic function $H_n(\mathbf{x})$ of degree $n$ we

$$\Delta\left(r^{2-d}H_n\left(\frac{\mathbf{x}}{r^2}\right)\right) = 0 \tag{10.21}$$

This follows from

$$\begin{aligned} r^{2-d}H_n\left(\frac{\mathbf{x}}{r^2}\right) &= r^{2-d-2n}H_n\left(\mathbf{x}\right) \\ &= r^{2-d-n}Y_n \end{aligned} \tag{10.22}$$

But by the last line of Eq. 10.15, with $n$ replaced by$2-d-n$,

$$\begin{aligned} \Delta\left(r^{2-d-n}Y_n\right) &= r^{-d-n}\left((2-d-n)\big(\cancel{2-d}-n+\cancel{(d-2)}\big)-L^2\right)Y_n \\ &= r^{-d-n}\left(n(n-2+d)\big)-L^2\right)Y_n \\ &= 0 \end{aligned} \tag{10.23}$$

and the last identity folllows from Eq. 10.16.

The functions describes multipoles: Harmonic functions that decay polynomially at infinity (see section 10.8.2). For example, in $d=3$ the dipole potential is

$$H_1(\mathbf{x}) = \mathbf{d} \cdot \mathbf{x} \Longleftrightarrow \frac{1}{r}H_1\left(\frac{\mathbf{x}}{r^2}\right) = \frac{\mathbf{d} \cdot \mathbf{x}}{|\mathbf{x}|^3} \tag{10.24}$$

Similarly, for $\mathbf{Q}$ traceless symmetric $3 \times 3$ matrix, the quadrupole potential is

$$H_2(\mathbf{x}) = \mathbf{x} \cdot \mathbf{Q}\mathbf{x} \Longleftrightarrow \frac{1}{r}H_2\left(\frac{\mathbf{x}}{r^2}\right) = \frac{\mathbf{x} \cdot \mathbf{Q}\mathbf{x}}{|\mathbf{x}|^5} \tag{10.25}$$

## 10.5 Poisson's equation

Poisson's equation is

$$\Delta\phi = -4\pi\rho \tag{10.26}$$

Clearly, if $\phi$ is a solution of the equation for some $\rho$, so is

$$\phi + \text{Harmonic} \tag{10.27}$$

So, Poisson's equation needs to be supplemented by boundary conditions that eliminated the freedom to add a harmonic function. It turns out that the appropriate boundary conditions in Euclidean space with dimensions $d \geq 3$, the right boundary condition is that $\phi$ vanishes at infinity.

Since Poisson's equation is linear, we can construct its solution by constructing $\rho$ from its point like elements

$$\rho(\mathbf{x}) = \int d\mathbf{y}\rho(\mathbf{y}-\mathbf{x})\delta(\mathbf{y}) \tag{10.28}$$

We call the solution $G$

$$\Delta G(\mathbf{x}) = \delta(\mathbf{x}) \tag{10.29}$$

and satisfies the boundary conditions, the Green function of the Laplacian. If you think of the linear operators $G$ and $\Delta$ as matrices and of Dirac $\delta$ as the unit matrix, then it is natural to think of the Green function as

$$G = \Delta^{-1}$$

By translation invariance of the Laplacian

$$\Delta G(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$$

and the solution of Poisson's equation

$$\phi(\mathbf{x}) = -4\pi \int G(\mathbf{x}-\mathbf{y})\rho(\mathbf{y})d\mathbf{y}, \tag{10.30}$$

## 10.6 Green's function

For unit point charge in 3-dimensions the electric field, by symmetry must be radial. Gauss law gives

$$\mathbf{E} = \frac{\mathbf{x}}{|\mathbf{x}|^3}, \quad \phi(x) = \frac{1}{|\mathbf{x}|} \tag{10.31}$$

It follows that the Green's function in $n = 3$ dimensions is

$$\Delta_x G(\mathbf{x}-\mathbf{y}) = \delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad G(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|} \tag{10.32}$$

### 10.6.1 Green function in arbitrary dimensions

The same method works in $d$ dimensions. Let

$$\mathbf{E} = \frac{1}{s_d}\frac{\mathbf{x}}{|\mathbf{x}|^d} = \frac{1}{s_d}\frac{\hat{\mathbf{r}}}{r^{d-1}}, \quad s_d = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}, \tag{10.33}$$

$s_d$ is the area of the d-dimensional unit sphere. This clearly satisfies Gauss law for unit source

$$\int_R \mathbf{E}\cdot dS_d = \int \underbrace{\nabla\cdot\mathbf{E}}_{=\delta(\mathbf{x})} dV_d = 1 \tag{10.34}$$

for any sphere of radius $R$. The source is a delta function. For $d > 2$ we have

$$\nabla\left(\frac{1}{r^{d-2}}\right) = -\frac{d-2}{r^{d-1}}\hat{\mathbf{r}}$$

it follows that the Green function is

$$G_d(r) = -\frac{1}{(d-2)s_d r^{d-2}}, \quad G_2(r) = \frac{1}{2\pi}\log\frac{r}{a}, \quad G_1(r) = \frac{r}{2} \tag{10.35}$$

$a > 0$ is an an arbitrary (length) scale factor.

In 1 and 2 dimensions the Green functions diverges at infinity and in $d \geq 2$ it diverges at the origin.

### 10.6.2 Volumes of d balls and spheres

Let

$$V_d(r) = v_d r^d, \quad S_d(r) = s_d r^{d-1}$$

be the volume and area of the ball and sphere in d dimensions. Since

$$V_d'(r) = S_d(r)$$

we see that

$$dv_d = s_d$$

Lets us find a recursion relation for the volumes. For $d \geq 1$ we have

$$\begin{aligned}
v_{d+2} &= \int_{|\mathbf{x}_{d+2}|\leq 1} d\mathbf{x}_d \\
&= \int_{|\mathbf{x}_d|^2+r^2\leq 1} d\mathbf{x}_d dx_{d+1} dx_{d+2} \\
&= 2\pi \int_{|\mathbf{x}_d|\leq 1} d\mathbf{x}_d \int_0^{\sqrt{1-|\mathbf{x}_d|^2}} r dr \\
&= \pi \int_{|\mathbf{x}_d|\leq 1} d\mathbf{x}_d (1-|\mathbf{x}_d|^2) \\
&= \pi v_d - \pi s_d \int_{r\leq 1} r^{d+1} dr \\
&= \pi\left(v_d - \frac{s_d}{d+2}\right) \\
&= \pi\left(1 - \frac{d}{d+2}\right) v_d \\
&= \frac{2\pi}{d+2} v_d
\end{aligned}$$

Sanity check:

$$v_3 = \frac{2\pi}{3} v_1 = \frac{4\pi}{3}$$

Here is an amusing observation about spheres: Evidently

$$v_{2d+1} = \frac{(2\pi)^d}{(2d+1)!!} v_1$$

$v_d \to 0$ super-exponentially.

**Exercise 10.9.** *Compute $s_d$ using the Gaussian integral*

$$\int d^d x e^{-x^2} = s_d \int_0^\infty r^{d-1} dr e^{-r^2}$$

## 10.7 Proof of the fundamental property of Harmonic functions

Let $G(x)$ be the Green function of the Laplacian in d-dimensions and $\phi$ Harmonic:

$$\begin{aligned} \Delta \left( G(x)\phi(x) \right) &= G\Delta\phi + 2\nabla\phi \cdot \nabla G + \phi\Delta\left(G\right) \\ &= 2\nabla\phi \cdot \nabla G + \phi(x)\,\delta(x) \end{aligned}$$

Integrate this identity on a ball at the origin. The last term (on the right) gives $\phi(0)$. The middle term can be organized so that one first integrates over directions at fixed radius and last over the radial direction. Since $G$ is a radial function we pull it to the left and write

$$\int_{|x|\le R} 2\nabla\phi \cdot \nabla G dV = s_d \int_0^R r^{d-1}\, dr\, G'(r) \underbrace{\int_{|x|=r} dS \cdot \nabla\phi}_{0\ by\ Gauss}$$

($\phi$ is Harmonic, the flux through any closed surface of $\nabla\phi$ vanishes so the integral on the right vanishes for any $r > 0$.) It remains to integrate the term on the left

$$\begin{aligned} \int_{|x|\le R} \Delta\left(\phi G\right)\, dV &= \int_{|x|=R} dS \cdot \nabla\left(\phi(\mathbf{x}) G(r)\right) \\ &= \int_{|x|=R} dS \cdot \Big( \underbrace{(\nabla\phi) G(R)}_{0\ by\ Gauss} + \phi(\mathbf{x})\, G'(R)\hat{\mathbf{r}} \Big) \\ &= G'(R) \int_{|x|=R} dS \cdot \hat{r}\, \phi(\mathbf{x}) \\ &= \frac{1}{s_d R^{d-1}} \int_{|x|=R} dS \cdot \hat{r}\, \phi(\mathbf{x}) \end{aligned}$$

and Eq. 10.35 for $G(R)$ has been used. This is precisely the average of $\phi$ over the sphere of radius $R$.

## 10.8 Stationary magnetic fields

The magnetic field is determined by Ampere's law and "Gauss" law for the magnetic field

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} + \dot{\mathbf{E}} = \frac{4\pi}{c}\mathbf{J}$$

In the case $\dot{\mathbf{E}} = 0$ this reduces to

$$\nabla \cdot \mathbf{B} = 0, \quad \underbrace{\nabla \times \mathbf{B} = \frac{4\pi}{c}\,\mathbf{J}}_{Ampere} \tag{10.36}$$

Ampere's law then implies $\nabla \cdot \mathbf{J} = 0$ (and then also $\dot{\rho} = 0$). To solve Ampere's differential equation we are free to use any gauge we please. In particular, in the Coulomb gauge:

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla \cdot \mathbf{A} = 0$$

Using the identity

$$\nabla \times (\nabla \times \mathbf{A}) = -\Delta\mathbf{A} + \nabla(\nabla \cdot \mathbf{A})$$

we can write Amper's equation as Poisson's equations for (the vector valued) $\mathbf{A}$

$$\Delta\mathbf{A} = -\frac{4\pi}{c}\mathbf{J}$$

**Remark 10.10** (Consistency)**.** *The equation is consistent with the gauge condition* $\nabla \cdot \mathbf{A} = 0$ *since* $\nabla \cdot \mathbf{J} = 0$.

### 10.8.1 Biot-Savart law

The vector valued version of Eq. 10.30 reduces solving Poisson's equations to integration. Namely,

$$\mathbf{A}(\mathbf{x}) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y} \tag{10.37}$$

By taking the curl of the identity we obtain $\mathbf{B}$ as an explicit integral over the current:

$$\begin{aligned}
\mathbf{B}(\mathbf{x}) &= \nabla \times \mathbf{A}(\mathbf{x}) \\
&= \frac{1}{c}\int \underbrace{\nabla_x\left(\frac{1}{|\mathbf{x}-\mathbf{y}|}\right)}_{Coulomb} \times \mathbf{J}(y)\,dy \\
&= \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{y}) \times (\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3}\,d\mathbf{y}
\end{aligned}$$

This is the Biot-Savart law.

**Exercise 10.11** (A straight line of current)**.** *Consider a cylindrically symmetric tube carrying constant current $I$ along the $z$-axis. Using the cylindrical symmetry of the problem and the integral version of Ampere's law show that*

$$\mathbf{B} = 2I\,\frac{\hat{\mathbf{z}}\times\mathbf{x}}{|\mathbf{x}|^2}$$

**Exercise 10.12** (Constant magnetic fields)**.** *The vector potential of a constant magnetic field is a linear vector values function and so of the form*

$$\mathbf{A} = \mathbf{a}\times\mathbf{x} + (\mathbf{b}\cdot\mathbf{x})\mathbf{c}$$

*Show that*

$$\mathbf{B} = 2\mathbf{a} + \mathbf{b}\times\mathbf{c}, \quad \nabla\cdot\mathbf{A} = \mathbf{c}\cdot\mathbf{b}$$

### 10.8.2 Magnetic dipole

The current associated with thin loop of radius $a$ in the $x-y$ plane carrying current $I$ is

$$\begin{aligned} 2\mathbf{J} &= (-y, x, 0)\, I\delta(x^2+y^2-a^2)\delta(z) \\ &= \tfrac{1}{2}(I\hat{\mathbf{z}}\times\nabla)\theta(a^2-x^2-y^2)\,\delta(z) \end{aligned}$$

and $\theta(x) = 1$ for $x > 0$ and 0 otherwise is the standard step function. Consider the limit $a \to 0$ and $I \to \infty$ so that that $Ia^2$ is fixed.

**Exercise 10.13** (Delta function)**.** *Show that*

$$\lim_{a\to 0} \frac{\theta(a^2-x^2-y^2)}{\pi a^2}\delta(z) = \delta(\mathbf{x})$$

The $a \to 0$ limit represents a point dipole, characterized by a vector

$$\mathbf{m} = \left(\frac{\pi^2 I a^2}{c}\right)\hat{\mathbf{z}}$$

Ampere equation takes the form

$$(\nabla\times\mathbf{B})(\mathbf{x}) = 4\pi(\mathbf{m}\times\nabla)\delta(\mathbf{x})$$

To find $\mathbf{B}$ we could plug the source into Biot-Savart. However, this is not much simpler then retracing the derivation. For **A** we find

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \int d\mathbf{y}\,\frac{(\mathbf{m}\times\nabla)_y\delta(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \\
&= \int d\mathbf{y}\left(\underbrace{(\mathbf{m}\times\nabla)_y\left(\frac{\delta(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right)}_{bdry\ term} - (\mathbf{m}\times\nabla)_x\frac{\delta(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\right) \\
&= -(\mathbf{m}\times\nabla)_x\int d\mathbf{y}\,\frac{\delta(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \\
&= -(\mathbf{m}\times\nabla)\,\frac{1}{|\mathbf{x}|} \\
&= \frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}
\end{aligned}$$

To compute **B** we need a version of the vector identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$$

where $\mathbf{a}=\nabla$ is a differential operator,$\mathbf{b}=\mathbf{m}$ a fixed vector and $\mathbf{c}=\mathbf{x}r^{-3}$ a vector valued function. Reflection shows that the right form is

$$\nabla\times(\mathbf{m}\times\mathbf{c}) = \mathbf{m}(\nabla\cdot\mathbf{c}) - (\mathbf{m}\cdot\nabla)\mathbf{c}$$

From the solution of the Coulomb problem we know that

$$\nabla\cdot(\mathbf{x}r^{-3}) = 4\pi\delta(\mathbf{x})$$

Hence

$$\mathbf{B}(\mathbf{x}) = 4\pi\mathbf{m}\delta(\mathbf{x}) - (\mathbf{m}\cdot\nabla)\left(\frac{\mathbf{x}}{r^3}\right)$$

It follows that the magnetic field of a dipole is

$$\mathbf{B} = \frac{-\mathbf{m}+3(\mathbf{m}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}}{|\mathbf{x}|^3} + 4\pi\mathbf{m}\delta(\mathbf{x})$$

**Exercise 10.14.** *Verify all steps.*

**Remark 10.15** (Singularity)**.** *The magnetic field has a bad (non-integrable) singularity at the origin. One way to see this is to consider the total flux through the origin. Take the plane oriented with* **m** *through the dipole. The flux through such a plane is*

$$\begin{aligned}
\mathbf{B}\cdot\mathbf{m} &= -\frac{(\mathbf{m}\cdot\mathbf{m})(\mathbf{x}\cdot\mathbf{x}) - 3(\overbrace{(\mathbf{m}\cdot\mathbf{x})}^{=0})^2}{|\mathbf{x}|^{\mathbf{5}}} + 4\pi\mathbf{m}^2\delta(\mathbf{x}) \\
&= -\frac{(\mathbf{m}\times\mathbf{x})^2}{|\mathbf{x}|^{\mathbf{5}}} + 4\pi\mathbf{m}^2\delta(\mathbf{x})
\end{aligned}$$

*The first term has an non-integrable singularity at the origin. At the same time, we know that the total flux through any surface must be zero*

**Exercise 10.16** (Vanishing flux)**.** *Show that the total flux through any such plane at distance $\varepsilon$ from the origin vanishes.*

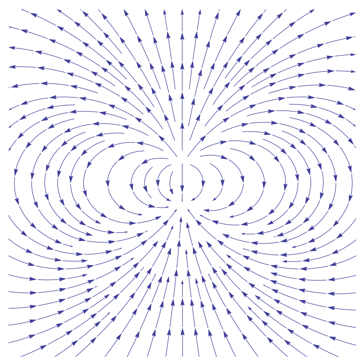


Figure 10.2: Dipole field

## 10.9 Dirac monopoles

Dirac monopoles were invented, by Dirac of course, to explain the quantization of charge: The charge of the proton is exactly minus that of the electron, in contrast with, say, their their mass ratio which does not look like a simple fraction. Dirac realized that if there was even a single monopole of magnetic charge $e_m$ anywhere in the universe, say, behind Andromeda, then charged quantization will be a consequence of quantum mechanics: The electric charge of any quantum particle $e$ will be constrained by

$$\frac{2e_m e}{\hbar c} \in \mathbb{Z}$$

This is Dirac charge quantization. This Dirac quantization can be viewed as a consequence of the Aharonov-Bohm effect. We look for a vector potential whose flux is the monopole charge $e_m$, and with nice Coulombic field

$$\nabla \times \mathbf{A} = e_m \frac{\mathbf{x}}{|\mathbf{x}|^3} \tag{10.38}$$

There is no smooth $\mathbf{A}$ that solves nevertheless. In spherical coordinates the three equations for $\mathbf{A}$ are

$$(\nabla \times \mathbf{A})^r = \frac{1}{\sqrt{g}}(\partial_\theta A_\phi - \cancel{\partial_\phi A_\theta}) = \frac{e_m}{r^2}, \quad g = r^4 \sin^2\theta \tag{10.39}$$

and

$$(\nabla \times \mathbf{A})^\theta = \frac{\cancel{\partial_\phi A_r} - \partial_r A_\phi}{\sqrt{g}} = 0, \quad (\nabla \times \mathbf{A})^\phi = \frac{\cancel{\partial_r A_\theta} - \cancel{\partial_\theta A_r}}{\sqrt{g}} = 0 \tag{10.40}$$

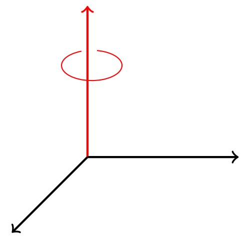


Figure 10.3: An infinitesimal loop around the positive z-axis carries flux of $4\pi e_m$ while the same loop around the negative z-axis carries zero flux.

The cancelled term is a consequence of choosing $A_r = A_\theta = 0$ which is suggested by symmetry. As we shall see, this choice is good enough.

Eq. 10.39 reduces to

$$\partial_\theta A_\phi = e_m \sin\theta \Longrightarrow A_\phi = e_m(1-\cos\theta) \tag{10.41}$$

and we have chosen the integration constant so that $A_\phi = 0$ on the positive z-axis $\theta = 0$. Since $A_\phi$ is independent of $r$ we also satisfy Eq. 10.40.

It seems that we have successfully solved Eq. 10.38 and the solution appears to be nice. But we know this can not be, since ah honest $\mathbf{A}$ implies $\nabla\cdot\mathbf{B} = 0$ and no monopoles. Indeed, $A_\phi \neq 0$ on the negative z-axis, where $\theta = \pi$, hides a singularity. Spherical coordinates are singular when $\sin\theta = 0$. A smooth vector field must therefore have $A_\phi = 0$ whenever $\sin\theta = 0$. The vector field we found is nice on the positive z-axis but singular on the negative z-axis. The singularity is called the Dirac string. You can see this from stokes for a small loop around the z-axis. If you integrate $\oint \mathbf{A}\cdot d\ell$ around the loop one finds

$$\oint A_\phi d\phi = 4\pi e_m \tag{10.42}$$

This can be given two interpretation: As the total flux through the sphere minus a tiny hole near the north pole, or as minus the total flux through the hole. $4\pi e_m$ is indeed the total flux of a monopole. Hence the Dirac string can be thought of as the solenoid the feeds the flux of the monople. Now, how shall we think of this Dirac string. On the one hand it is a gauge dependent property. We could have chosen our coordinate system so that the string would go from the origin to infinity along any direction we wish. On the other hand the Dirac string carries magnetic field with flux $4\pi e_m$. Magnetic fields are physical!

The problem would go away if we could use quantum mechanics to argue that there are certain invisible magnetic fluxes. Indeed, this is what the Aharonov-Bohm effect tells us: In an interference experiment involving a flux tube, one can not distinguish no flux from a flux with integer number of quantum flux quanta. This says that the Dirac string becomes invisible if

$$4\pi e_m = \text{Integer} \times \Phi_0, \quad \Phi_0 = \frac{hc}{e} \tag{10.43}$$

We can rearrange this as

$$e = \text{Integer} \times \frac{\hbar c}{2e_m} \tag{10.44}$$

This is what Dirac set up to show: All electric charges must be an integer multiple of a basic charge. I'd like now to derive Dirac result without appealing

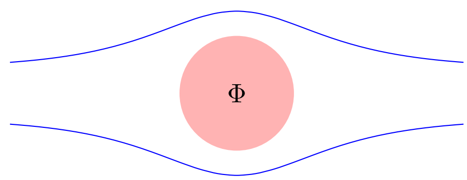



Figure 10.4: The interefrence pattern from a flux tube is periodic in the flux with period $\Phi_0 = \frac{hc}{e}$

to the Aharonov-Bohm effect. This argument is more mathematical, but you will see it often.

We can get rid of Dirac string by doing something that geographers do: Plotting earth on one sheet of paper leads to singularities at the pole. The problem can be avoided by plotting the north hemisphere on one sheet and the south hemisphere on the another.

On the half space $z \leq 0$ we us the gauge

$${A^N}_\phi = e_m(1 - \cos\theta) \tag{10.45}$$

which has a Dirac string along $z \geq 0$. On the half space $z \geq 0$ we us the gauge

$${A^N}_\phi = e_m(-1 - \cos\theta) \tag{10.46}$$

which has a Dirac string along $z \leq 0$.

We now need to "glue" the two half spaces by a gauge transformation. It is here that quantum mechanics enters. We need three facts

- A unitary transformation of the state $|\psi\rangle$ and operators $W$ gives an equivalent description of the system

$$(|\psi\rangle, W) \Longleftrightarrow \left(U|\psi\rangle, UWU^\dagger\right) \tag{10.47}$$

- Charged particles couple to the electromagnetic field through minimal coupling

$$-i\hbar\partial_j - \frac{e}{c}A_j \tag{10.48}$$

- A gauge transformation is unitary $U$ that is a function of the coordinates.

It follows that a gauge transformation affects minimal coupling

$$-i\hbar\partial_j - \frac{e}{c}A_j \Longrightarrow U\left(-i\hbar\partial_j - \frac{e}{c}A_j\right)U^\dagger = -i\hbar\partial - A_j + \underbrace{i\frac{\hbar c}{e}\partial_j \log U}_{\text{pure gauge}} \tag{10.49}$$

We can try to glue the two half space by taking $U = e^{-in\phi}$. This is possible provided

$$A^N{}_\phi - A^S{}_\phi = i\frac{\hbar c}{e}\partial_j \log U \partial_\phi \phi = n\frac{\hbar c}{e} \tag{10.50}$$

Comparing with Eqs. 10.45, 10.46 we see that

$$A^N{}_\phi - A^S{}_\phi = n\frac{\hbar c}{e} = 2e_m \tag{10.51}$$

We recovered Eq. 10.44. This gives Dirac quantization rule.

## 10.10 Application to geometry

### 10.10.1 Vector fields in 3D: Source and vorticity

Given a vector field $\mathbf{V}$ in 3 dimensions, its source $\rho$ and vorticity $\boldsymbol{\omega}$ are defined by:

$$\nabla \cdot \mathbf{V} = 4\pi\rho, \quad \nabla \times \mathbf{V} = 4\pi\boldsymbol{\omega}$$

Basic facts are:

- Vorticity is sourcesless
$$\nabla \cdot \boldsymbol{\omega} = 0$$
- Radial vector fields are vorticity free.
- $\mathbf{x}$ is vector field with uniform source, $\nabla \cdot \mathbf{x} = 3$.

The converse is also true: The sources $\rho$ and $\boldsymbol{\omega}$ with $\nabla \cdot \boldsymbol{\omega} = 0$ determine the field $\mathbf{V}$. By linearity, we can decompose the problem of constructing $\mathbf{V}$ into two problems:

$$\mathbf{V} = \mathbf{E} + \mathbf{B}$$

where $\mathbf{E}$ is irrotational (conservative)

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad \nabla \times \mathbf{E} = 0$$

and $\mathbf{B}$ is sourceless

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = 4\pi\boldsymbol{\omega}$$

As we have seen, the equations for $\mathbf{E}$ and $\mathbf{B}$ are solved by the same technique.

### 10.10.2 Linking number

Suppose you have two loops $\gamma_1$ and $\gamma_2$ in space and you want to know if they link. Imagine that the loop $\gamma_1$ carries a unit current. Then, if the loop $\gamma_2$ links $n$ times the loop $\gamma_1$ we have

$$\int_{\gamma_2} \mathbf{B}(\mathbf{x_2}) \cdot d\mathbf{x}_2 = \frac{4\pi n}{c}$$

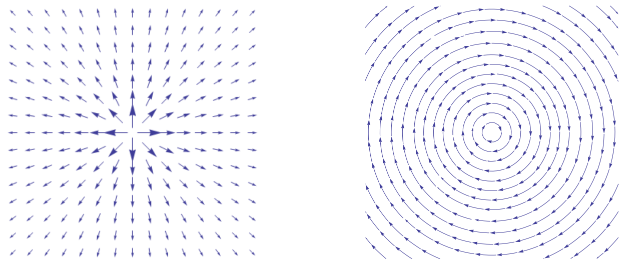


Figure 10.5: Left: An irrotational field with a source at the origin. A sourceless field with vorticity along the z-axis

Now plug **B** from the solution of Poisson's equation to get

$$\int_{\gamma_2}\int_{\gamma_1}\frac{d\mathbf{x}_2\cdot(\mathbf{x_2}-\mathbf{x_1})\times d\mathbf{x_1}}{|\mathbf{x_2}-\mathbf{x_1}|^3}=4\pi n \tag{10.52}$$

If $n \neq 0$ the loops link. The converse is, however, not always true.

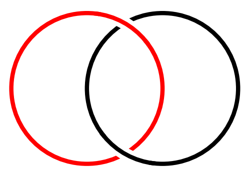


Figure 10.6: Linking circles. The linking number, here 1, can be computed from Eq. 10.52

# Chapter 11

# Electromagnetic waves

## 11.1 Maxwell's equations in the Lorenz gauge

The inhomogeneous Maxwell equations are:

$$\partial^\mu F_{\mu\nu} = -\frac{4\pi}{c} j_\nu$$

The homogeneous Maxwell's equations are automatically satisfied if we write $F$ in terms of $A$. Hence, all of Maxwell's equations are simply encoded in

$$\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = -\frac{4\pi}{c} j_\nu \tag{11.1}$$

Gauge freedom allows us to impose a gauge condition on $A$, and it is convenient to pick the Lorenz gauge[1]

$$\partial^\mu A_\mu = 0 \tag{11.2}$$

Maxwell's equations in the Lorenz gauge are a set of decoupled wave equations for $A_\mu$ with source $j_\mu$

$$\partial^\mu \partial_\mu A_\nu = \Box A_\nu = -\frac{4\pi}{c} j_\nu \tag{11.3}$$

where

$$\Box = \partial^\mu \partial_\mu = -\frac{1}{c^2}\partial_{tt} + \Delta$$

is the D'Alambertian, the operator associated with the wave equation.

### 11.1.1 Ambiguity of the Lorenz gauge

Unlike the coulomb gauge, the Lorenz gauge does not fix $A_\mu$ uniquely. Indeed, if $A_\mu$ satisfies the Lorenz gauge so does

$$A_\mu \mapsto A_\mu + \partial_\mu \Lambda \tag{11.4}$$

[1] I shall show later that this is always possible.

provided $\Lambda$ satisfies the wave equation

$$\partial^\mu \partial_\mu \Lambda = 0 \tag{11.5}$$

## 11.2 Electromagnetic waves

In the absence of 4-currents and in the Lorenz gauge the potentials satisfy the wave equation

$$\Box A_\mu = 0 \tag{11.6}$$

**Remark 11.1** (Lorentz invariance)**.** *Since the Dalambertian, $\Box$ and the Lorenz gauge conditions are manifestly a Lorentz invariants, Lorentz transformations of electromagnetic waves are electromagnetic waves.*

### 11.2.1 Electric and Magnetic fields

Since the gauge fields are not measurable objects, it is useful to see that the fields too satisfy the wave equation with a source term. Taking the time derivative of Faraday law

$$\dot{\mathbf{B}} + c\nabla \times \mathbf{E} = 0 \Longrightarrow \ddot{\mathbf{B}} + c\nabla \times \dot{\mathbf{E}} = 0$$

Substituting Ampere law

$$-\dot{\mathbf{E}} + c\nabla \times \mathbf{B} = 4\pi \mathbf{J}$$

gives

$$\ddot{\mathbf{B}} = c\nabla \times (-c\nabla \times \mathbf{B} + 4\pi \mathbf{J}) = c^2 \Delta \mathbf{B} - c^2 \nabla \underbrace{(\nabla \cdot \mathbf{B})}_{=0} + 4\pi c \nabla \times \mathbf{J}$$

This leads to the wave equation with a source them that is proportional to $\nabla \times \mathbf{J}$:

$$\Box \mathbf{B} = -\frac{4\pi}{c} \nabla \times \mathbf{J}$$

In particular, when $\nabla \times \mathbf{J} = 0$ we get the free wave equation for the magnetic field.

Similarly for the electric field we have from Ampere's law

$$\ddot{\mathbf{E}} + c\nabla \times \dot{\mathbf{B}} = 4\pi \dot{\mathbf{J}}$$

Substituting Faraday's law and making use of Gauss law gives

$$\begin{aligned} \ddot{\mathbf{E}} &= -c^2 \nabla \times (\nabla \times \mathbf{E}) + 4\pi \dot{\mathbf{J}} = c^2 \Delta \mathbf{E} - c^2 \nabla (\nabla \cdot \mathbf{E}) + 4\pi \dot{\mathbf{J}} \\ &= c^2 \Delta \mathbf{E} - 4\pi \left( c^2 \nabla \rho - \dot{\mathbf{J}} \right) \end{aligned}$$

The electric field satisfies the wave equation with a source term proportional to $\nabla \rho - \dot{\mathbf{J}}/c^2$

$$\Box \mathbf{E} = 4\pi \left( \nabla \rho - \frac{\dot{\mathbf{J}}}{c^2} \right)$$

In the absence of source terms, the electric field satisfies the free wave equation. This was one of Maxwell's great discoveries, namely, that even in the absence of sources, the equations admit interesting wave-like solutions. This allowed him to interpret light coming from the stars and propagating in vacuum, as electromagnetic waves and eventually lead to the discovery of the radio.

## 11.3 Plane waves

Consider (the real part of)

$$A_\mu(x) = a_\mu e^{ik\cdot x}, \quad k\cdot x = k_\mu x^\mu \tag{11.7}$$

with $a_\mu$ a 4-vector of fixed amplitudes and $k_\mu = (-\omega/c, \mathbf{k})$ a fixed 4-wave vector. The associated field is

$$F_{\mu\nu} = i(k_\mu a_\nu - k_\nu a_\mu)e^{ik\cdot x}$$

Maxwell equations, Eq. (11.6), reduce to an algebraic equation for $k$ and a linear equation for $a_\nu$

$$(k\cdot k)\ a_\nu - k_\nu(k\cdot a) = 0 \tag{11.8}$$

By the Lorenz gauge condition, $(k\cdot a) = 0$ and hence also

$$k\cdot k = 0, \quad k\cdot a = 0$$

This says that $k$ is a light-like vector. The dispersion relation is

$$\omega = \pm c|\mathbf{k}|$$

To understand the Lorenz gauge condition, $k\cdot a = 0$, lets us orient the Euclidean frame so that the wave propagates in the z-direction. The light-like vector $k_\mu$ and the amplitude $a_\mu$ are then

$$k_\mu = \frac{\omega}{c}(-1,0,0,1), \quad a_\mu = \underbrace{(a_0, a_1, a_2, -a_0)}_{Lorentz\ gauge}$$

This is how a plane wave looks in the Lorenz gauge.

There is a remnant gauge freedom that allows us to choose $a_0$. The part proportional to $a_0$ is

$$a_0(1,0,0,-1)e^{i\omega(t-z/c)} = i\frac{ca_0}{\omega}\partial_\mu e^{i\omega(t-z/c)}$$

is a pure gauge. This allows us to set $a_0 = 0$ reducing the Lorenz gauge to the Coulomb gauge:

$$k_\mu = \frac{\omega}{c}(-1,0,0,1), \quad a_\mu = \underbrace{(0, a_1, a_2, 0)}_{Coulomb\ gauge} \tag{11.9}$$

In the Coulomb gauge, the amplitudes are orthogonal (in Euclidean space) to the direction of propagation.

### 11.3.1 Electric and magnetic fields

For plane waves

$$\mathbf{E} = -i\frac{\omega}{c}\mathbf{A}, \quad \mathbf{B} = -i\mathbf{k}\times\mathbf{A}$$

Since $\mathbf{k}\cdot\mathbf{A} = 0$ this implies that $\mathbf{E}$ and $\mathbf{B}$ are orthogonal and have equal magnitudes. $\mathbf{E}, \mathbf{B}$ and $\mathbf{k}$ form an orthogonal triad.

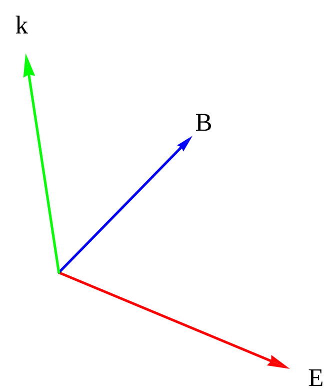



Figure 11.1: The triad of $\mathbf{E}, \mathbf{B}, \mathbf{k}$ for a plane wave. The wave propagates in the $\hat{\mathbf{k}} = \hat{\mathbf{E}}\times\hat{\mathbf{B}}$ direction

### 11.3.2 Doppler

Since $k\cdot x$ is a Lorentz scalar

$$k\cdot x = k'\cdot x'$$

Lorentz transformation of a plane wave is a plane wave. The wave has, in general, different wave vectors and amplitudes in different frames:

$$k'_\mu = \Lambda_\mu{}^\nu k_\nu, \quad a'_\mu = \Lambda_\mu{}^\nu a_\nu$$

#### Longitudinal Doppler

Consider a plane wave propagating in the z-direction, Eq. (11.9). Boosting the wave with rapidity $\phi$ in the *same* direction is the same as viewing the wave from an inertial frame boosted in the *opposite* direction. The associated Lorentz transformation is

$$\Lambda_0{}^3 = \Lambda_3{}^0 = -\sinh\phi, \quad \Lambda_0{}^0 = \Lambda_3{}^3 = \cosh\phi, \quad \Lambda_1{}^1 = \Lambda_2{}^2 = 1 \qquad (11.10)$$

The Lorentz transformation of the light-like vector $k_\mu = \omega(1,0,0,1)$ gives $k'_\mu = \omega'(1,0,0,1)$ where

$$\omega' = \omega(\cosh\phi + \sinh\phi) = \omega e^{\phi} = \omega\sqrt{\frac{1+\beta}{1-\beta}} \tag{11.11}$$

and we used the relation between rapidity and velocity

$$\beta = \tanh\phi \tag{11.12}$$

The Doppler shift is linear in the velocities for small speeds. $a_{1,2}$ are not affected by the boost.

**Transverse Doppler**

Consider, as before, a wave propagating in the z-direction, but a boost in the x-direction so that

$$\Lambda_0{}^1 = \Lambda_1{}^0 = -\sinh\phi, \quad \Lambda_0{}^0 = \Lambda_1{}^1 = \cosh\phi, \quad \Lambda_2{}^2 = \Lambda_3{}^3 = 1 \tag{11.13}$$

The wave vector $k_\mu = \omega(1,0,0,1)$ is transformed to a light like wave vector $k'_\mu = \omega(\cosh\phi, -\sinh\phi, 0, 1)$ in the $x-z$ plane. The new frequency is

$$\omega' = \omega\cosh\phi = \omega\,\gamma$$

This is quadratic in the velocities for small speeds.

## 11.4 Polarization

### 11.4.1 Amplitude and phase

Scalar plane waves are simply characterized by their frequency $\omega$, wave vector $\mathbf{k}$, amplitude and phase. Electromagnetic waves, being vector valued, are more complicated. In addition to the amplitude and phase they are also characterized by their polarization.

The electric field of an electromagnetic plane wave propagating is the real part of

$$\mathbf{E}_0\, e^{i\phi}, \quad \phi = \mathbf{k}\cdot\mathbf{x} - \omega t \tag{11.14}$$

The amplitude, $\mathbf{E}_0$, is a *complex* vector in the plane perpendicular to $\mathbf{k}$ since $\mathbf{k}\cdot\mathbf{E}_0 = 0$. $\mathbf{E}_0$ therefore has 4 real amplitudes. What is their physical interpretation? Suppose we scale

$$\mathbf{E}_0 \mapsto \lambda\mathbf{E}_0, \quad \lambda = |\lambda| e^{i\gamma} \tag{11.15}$$

One would then say that the amplitude has been scaled by $|\lambda|$ and the phase shifted by $\gamma$. We have now identified 2 of the 4 parameters in $\mathbf{E}_0$: An amplitude and a phase. It remains to identify the remaining two parameters hidden in $\mathbf{E}_0$. We are now in a situation that is reminiscent of quantum mechanics: The wave function is a complex vector with an equivalence relation by an overall complex number.

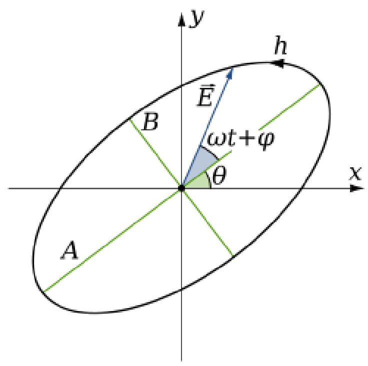



Figure 11.2: Four Stokes parameters describe elliptically polarized light. Three numbers identify the size of the ellipse, its tilt to the axes, its eccentricity. A fourth number gives the purity (the coherence) of the light. The plane of the ellipse is perpendicular to the direction of propagation **k**.

### 11.4.2 Polarization

Let $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ denote orthogonal unit vectors in the plane perpendicular to **k**. Let me introduce basis vectors, with Dirac ket notation

$$|z_{\pm}\rangle = \frac{\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}}{\sqrt{2}} \tag{11.16}$$

The states are normalized and orhtogonal

$$\langle z_{\pm}|z_{\pm}\rangle = 1, \quad \langle z_{\pm}|z_{\mp}\rangle = 0 \tag{11.17}$$

The complex vector $\mathbf{E}_0$ can be represented as

$$\mathbf{E}_0 \Leftrightarrow E_+ |z_+\rangle + E_- |z_-\rangle = |E\rangle \tag{11.18}$$

We are interested in the degrees of freedoms that remain after we identify $\mathbf{E}_0$ with $\lambda\mathbf{E}_0$. This is precisely the equivalence relation we have in quantum mechanics for qubits. Quantum mechanics provides us with a procedure for factoring out the normalization and the overall phase in of a quantum state. This is the density matrix representation of quantum states:

$$\rho = \frac{|E\rangle\langle E|}{\langle E|E\rangle} = \frac{1}{|E_+|^2 + |E_-|^2}\begin{pmatrix} |E_+|^2 & E_+E_-^* \\ E_+^*E_- & |E_-|^2 \end{pmatrix} \tag{11.19}$$

which does not care about the overall normalization and phase. Since $Tr\rho = 1$ while $\det\rho = 0$ the two eigenvalues of $\rho$ are 1 and 0: $\rho$ is a projection

$$\rho^2 = \rho \tag{11.20}$$

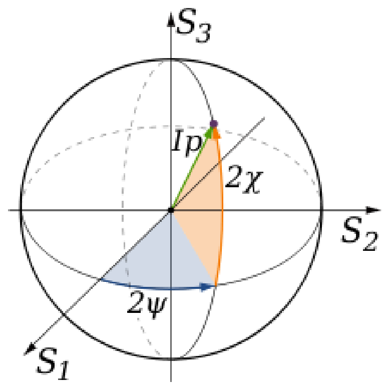



Figure 11.3: The Poincare sphere associates with every point on the sphere a polarization. The north and south poles represent right and left circularly polarized light and the equator with linearly polarized light.

### 11.4.3 Poincare sphere

Any $2 \times 2$ hermitian matrix of unit trace can be written as

$$\rho = \tfrac{1}{2}\left(1 + \mathbf{s}\cdot\sigma\right) = \frac{1}{2}\begin{pmatrix} 1+s_3 & s_1+is_2 \\ s_1-is_2 & 1-s_3 \end{pmatrix} \tag{11.21}$$

where $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{11.22}$$

From Eq.( 11.21)

$$4 \det \rho = 1 - \mathbf{s}\cdot\mathbf{s} \tag{11.23}$$

so $\rho$ is a projection if $\mathbf{s}$ is a unit vector $\hat{\mathbf{s}}$. This identify the remaining two parameter with points on the unit sphere. The physical meaning of these degrees of freedom is polarization. It is interesting that these natural degrees of freedom are parametretized by the sphere.

**Exercise 11.2.** *Show that antipodal points on the Poincare sphere represent orthogonal states in the sense that* $\rho_{\hat{s}} \cdot \rho_{-\hat{s}} = 0$.

### 11.4.4 Circular polarization

The north poles of the Poincare sphere correspond to $|z_+\rangle$ so that

$$\mathbf{E} \propto Re\big((\mathbf{x} + i\mathbf{y})e^{i\phi}\big) = \mathbf{x}\cos\phi - \mathbf{y}\sin\phi \tag{11.24}$$

As $\phi$ increases from 0 to $2\pi$ the vector $\mathbf{E}$ describes a circle in the x-y plane which is turning counter clockwise. At a fixed $t$ and as a function of $z$ the field rotates counter-clockwise like a right handed screw and so is called right circularly polarized.

**Exercise 11.3** (South pole)**.** *Show that the south pole represent left circular polarization.*

### 11.4.5 Linear polarization

The equator is $s_3 = 0$. This implies that $|E_+| = |E_-|$ and so the states on the equator are parameterized by $\psi$:

$$(e^{-i\psi/2}, e^{i\psi/2})/\sqrt{2} \tag{11.25}$$

The corresponding $\mathbf{E}_0$ is a real vector in the plane:

$$\begin{aligned}\mathbf{E}_0 &= \tfrac{1}{2}(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{-i\psi/2} + \tfrac{1}{2}(\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i\psi/2} \\ &= \hat{\mathbf{x}}\cos(\psi/2) + \hat{\mathbf{y}}\sin(\psi/2)\end{aligned}$$

As $\psi$ increases from 0 to $2\pi$ this describes a line element in the x-y plane at angle $\psi/2$ to the $x$ axis. $\psi = 0$ corresponds to $\hat{\mathbf{x}}$ polarized wave and $\psi = \pi$ to $\hat{\mathbf{y}}$ polarized wave.

### 11.4.6 Stokes parameters

Since we are oblivious to the overall phase, we may write $(E_+, E_-) = (\cos\chi, e^{i\psi}\sin\chi)$, with $\cos\chi \geq 0$, i.e. $0 \leq \chi \leq \pi/2$ and $0 \leq \psi < 2\pi$. The corresponding points on the unit sphere are

$$\begin{aligned}s_3 &= \cos^2\chi - \sin^2\chi = \cos 2\chi, \\ s_1 + is_2 &= 2e^{-i\psi}\cos\chi\sin\chi = e^{-i\psi}\sin 2\chi\end{aligned}$$

This makes $2\chi$ and $\psi$ the standard spherical coordinates.

### 11.4.7 Partially polarized light

The discussion so far addressed what are commonly known as coherent waves: The waves are given by a function. In practice, one often encounters situations where the light one gets is from and ensemble of different sources and one has only statistical information about the plane waves. Such waves are called incoherent. One way to model incoherence is as a statistical average

$$\mathbf{E}_0 = \sum p_j \mathbf{E}_j, \quad p_j \geq 0,$$

where $\mathbf{E_j}$ represent independent (normalized) light sources, namely

$$\langle (\mathbf{E}_j)_a (\mathbf{E_k})_b \rangle = 0 \quad j \neq k, \forall a, b$$

In the case that all these sources emit plane waves all sharing the same direction $\hat{\mathbf{k}}$ the polarization of the mixture is naturally defined as the mixture of polarizations

$$\rho = \sum p_j \rho_j \tag{11.26}$$

**Exercise 11.4.** *Show that*

$$\langle \rho \rangle = \tfrac{1}{2}\left(1 + \langle \mathbf{s} \rangle \cdot \boldsymbol{\sigma}\right) \tag{11.27}$$

It is still true that $Tr\rho = 1$. But now $\langle \rho \rangle$ need not be a projection since the averages of a unit vectors is shorter, in general, than a unit vector: $|\langle \mathbf{s} \rangle| \leq 1$; the vector lies in the unit ball. The light we get from the sun is completely unpolarized. It is associated with $\mathbf{s} = 0$, the center of the Poincare ball.

**Remark 11.5** (Combing a tennis ball)**.** *One amusing, essentially topological, property of the transverse nature of electromagnetic waves is that it is not possible to have a fully spherically symmetric electromagnetic wave. The point is that a spherical wave, with* $\mathbf{k}$ *pointing radially, has* $\mathbf{E}$ *tangent to the sphere. It is a basic fact in topology that any vector field on the sphere must vanish at (at least) two points. The field can not be "the same" everywhere.*

## 11.5 The wave equation

So far, we have discussed plane wave solutions of the electromagnetic wave equation. We now turn to investigating the wave equation in general.

### 11.5.1 The wave equation in one dimension

The one dimensional wave equation, for a scalar field $\phi$, in light-cone coordinates $u = x^1 - x^0$, $v = x^1 + x^0$ takes the form

$$\Box\phi = 4\partial_{uv}\phi = 0$$

The general solution is

$$\phi(u, v) = f(u) + g(v)$$

with (essentially) arbitrary $f$ and $g$. $f$ describes a wave rigidly propagating to the right at speed $c$ and $g$ a wave rigidly propagating to the left at speed $c$.

The functions $f$ and $g$ can be determined by the initial (Cauchy) data

$$\phi_0(x^1 = x, x^0 = 0) = f(x) + g(x), \quad \dot{\phi}_0(x^1 = x, x^0 = 0) = -\big(f'(x) - g'(x)\big)$$

This allows for reconstructing $f$ and $g$ from the initial data by integration. One can verify that, the solution in terms of the initial data, is

$$\phi(x, t) = \frac{1}{2}\left(\phi_0(x + ct) + \phi_0(x - ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{\phi}_0(y)dy \tag{11.28}$$

### 11.5.2 Waves with Gaussian waists

A plane wave is an idealization: The field extends all the way to infinity in all directions and has infinite power. Let us now consider a model of a narrow pencil of light which has a finite width and power.

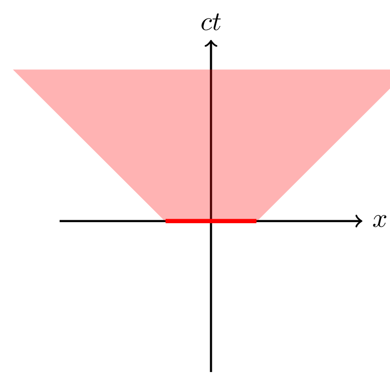



Figure 11.4: If the initial data, $\phi_0$ and $\dot{\phi}_0$ are localized in the red interval, the solution at later times lives in the forward light cone of the initial data. This is called the domain of influence of the initial (red) data. This statement holds in any inertial frame.

To do that it is convenient to write the wave equation in 3+1 dimensions as

$$(\partial_{uv} + \bar{\partial}\partial)\phi(u, v, z, \bar{z}) = 0, \quad z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2$$

where $u, v$ are light cone coordinates, $v = x^3 - x^0,\ u = x^3 + x^0$, where $x^0 = ct$.

A monochromatic wave with a Gaussian waist is a model of laser beam.

Consider a monochromatic wave with frequency $\omega = c/\lambda$ and the form

$$\phi(u, v, z, \bar{z}) = c(u)e^{-c(u)z\bar{z}}e^{iv/\lambda}$$

This ansatz solves the wave equation provided the function $c(u)$ satisfies

$$\lambda c^2(u) - ic'(u) = 0.$$

The solution of this equation has an integration constant giving for $c(u)$

$$c(u) = \frac{1}{\ell^2 - i\lambda u}$$

$\ell$ is the minimal waist of the beam at $u = 0$, and the waist disperses with the law

$$(\ell^4 + \lambda^2 u^2)^{1/4}$$

The waist broadens with the root of $u$, like diffusion.

**Exercise 11.6.** *A narrow red laser beam of diameter 1 mm and wavelength $\lambda = 700nm$ is directed at the moon. What is the diameter of the beam on the moon.*

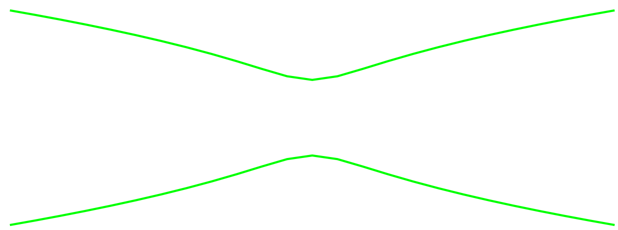


Figure 11.5: The waist of a Gaussian beam as function of $z + ct$.

## 11.6 Green's function for the wave equation

We can use the linearity of the wave equation to represent the solutions of the wave equations with a source, Eq.11.3, as the integral,

$$A_\mu(x) = \int dy j_\mu(y) G_4(x - y) \tag{11.29}$$

where $G_{d+1}$ is the Green function in space-time, i.e. a solution of the equation

$$\Box G_{d+1}(x) = \delta^{d+1}(x), \quad \delta^{d+1}(x) = \prod_\mu \delta(x^\mu) \tag{11.30}$$

Eq. 11.30 alone does not fix a unique is not unique $G_{d+1}$, as we can add to $G_{d+1}$ any solution of the free wave equation. Any solution $G_{d+1}$ satisfies the free wave in the past $t < 0$. We can therefore subtract from $G_{d+1}$ the free wave that agrees with the free wave in the past to have a solution with $G_{d+1}$ in the past. We can repeat the procedure for different rest frames and then we arrive at the conclusion that we can impose on $G_{d+1}$ the condition that it vanishes outside the forward light cone. It turns out that $G_{d+1}$ has different qualitative properties depending on whether $d$ is even or odd. Thus although we mostly care about $d = 3$, we shall consider the general case.

### 11.6.1 Conservation law

For $t \neq 0$, $G$ solves the free wave equation. Integration on a fixed time slice in $d + 1$ space-time

$$c^2 \int d^d x\, \partial_{tt} G = \int d^d x \Delta G = \int d^d x \nabla \cdot \nabla G = \int d^{d-1} x \nabla G = 0$$

It follows that

$$\int d^d x\, \partial_t G$$

is a conserved quantity for $t > 0$ (and trivially so for $t < 0$). It has a jump at $t = 0$:

$$\int d^d x\, \partial_t G = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

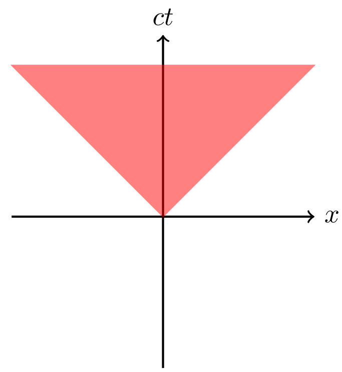



Figure 11.6: $G_{d+1} = 0$ outside the forward light cone. In particular $G_{d+1} = 0$ for $x_\mu x^\mu > 0$.

This is seen by integrating Eq. 11.30 on the space-time slice $-\epsilon < t < \epsilon$.

### 11.6.2 Recursion relation for the Green function

To find $G_4$ we shall construct a recursion relation that relates $G_4$ with $G_2$. $G_2$ can be computed explicitly. We shall now derive this recursion relation[2] that relates $G_{d+1}$ to $G_{d-1}$.

The starting observation is the obvious identity

$$\delta^d(x_0, \dots, x_d) = \int dx^{d+1} \delta^{d+1}(x_0, \dots, x_{d+1}) \tag{11.31}$$

The identity has the interpretation that a point source in $d$ space-time is a section of a line-source in $d+1$ space-time dimensions.

To proceed, we also observe that since the D'Alambertian is Lorentz invariant, it is natural to look for $G_{d+1}$ which is a function of the interval for $t > 0$

$$s = -x_\mu x^\mu$$

We can figure out $G_d$ from $G_{d+1}$ by superposition:

$$G_d(s) = \int_{-\infty}^{\infty} dx_{d+1} G_{d+1}(s - x_{d+1}^2)$$

[2] I have learned this recursion relation from Amos Ori.

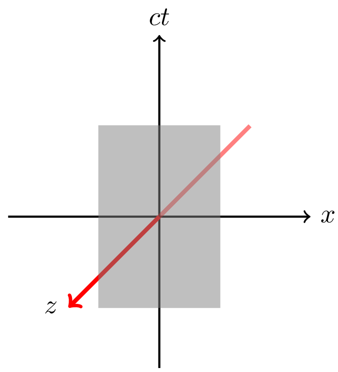



Figure 11.7: A point source is 2 dimensions is a two dimensional section of a line source in 3.

Iterating this relation one more time gives

$$\begin{aligned} G_d(s) &= \int_{-\infty}^{\infty} dx_{d+1} \int_{-\infty}^{\infty} dx_d G_{d+2}(s - x_d^2 - x_{d+1}^2) \\ &= \pi \int_0^{\infty} d(r^2) G_{d+2}(s - r^2) \\ &= \pi \int_{-s}^{\infty} G_{d+2}(-r) dr \\ &= \pi \int_{-\infty}^{s} G_{d+2}(r) dr \end{aligned}$$

Differentiating gives the sought after recursion relation

$$\frac{dG_d(s)}{ds} = \pi G_{d+2}(s) \tag{11.32}$$

So if you know the green function in 1 dimension, you can find it for all odd dimensions by differentiation, and if you know it for 2 dimensions you get it for all even dimensions.

### 11.6.3 Even space dimensions

In the case $d = 0$ the equation for the Green function degenerates to an ODE

$$\frac{d^2 G_1}{dt^2} = c^2 \delta(ct) \tag{11.33}$$

Integrating once, taking into account the light-cone condition, gives

$$\frac{dG_1}{dt} = c\theta(t) \tag{11.34}$$

Integrating once more

$$G_1(s) = ct\theta(t), \quad s = (ct)^2 \tag{11.35}$$

For positive times we get

$$G_{0+1}(s) = \sqrt{s} \tag{11.36}$$

From the recursion relation we then get for $G_{2+1}$ in the forward light cone

$$G_{2+1}(s) = \frac{1}{2\pi\sqrt{s}} \tag{11.37}$$

In even space dimensions $G_{d+1}$ fill the forward light cone. The Green function does not quite satisfy the Huygens principle as the wave does not live on the boundary of the light-cone, but rather fills it up. This is a feature of all even spatial dimensions.

### 11.6.4 Odd space dimensions

In the case $d = 1$ we can make use of the light-cone coordinates to find $G$. The equation for $G$ is

$$4\frac{\partial^2 G_{1+1}}{\partial u \partial v} = 2\delta(u)\delta(v) = \delta(ct)\delta(x) \tag{11.38}$$

Integrating gives

$$G_{1+1} = \frac{1}{2}\theta(u)\theta(v) = \frac{1}{2}\theta(t)\theta(s) \tag{11.39}$$

In summary

$$G_{1+1}(x) = \begin{cases} \frac{1}{2}\theta(s) & \text{in the forward light cone} \\ 0 & \text{otherwise} \end{cases} \tag{11.40}$$

(Note that once again the solution fills the light cone.) From the recursion relation we now find for $G_{3+1}$

$$G_{3+1}(s) = \begin{cases} \frac{1}{2\pi}\delta(s) & \text{in the forward light cone} \\ 0 & \text{otherwise} \end{cases} \tag{11.41}$$

Note that $G_{3+1}$ lives on the surface of the light cone. This is the Huygens principle. This continues to hold in all higher odd dimensions.

**Remark 11.7.** $G_{d+1}$ *is often written in the equivalent form*

$$G_{>}^{3+1}(x) = \theta(t)\frac{\delta(|\mathbf{x}| - ct)}{|\mathbf{x}|}, \quad x = (ct, \mathbf{x}), \tag{11.42}$$

**Exercise 11.8.** *Show this.*

## 11.7 Coulomb gauge

The Coulomb gauge, aka the radiation gauge, aka the transverse gauge is:

**Theorem 11.9** (Coulomb gauge)**.** *It is always possible to choose the vector potential* $A_\mu = (-\Phi, \mathbf{A})$ *so that*

$$\nabla \cdot \mathbf{A} = 0, \quad \Delta\Phi = -4\pi\rho \tag{11.43}$$

$\Phi$ *is the solution of Poisson's equation the gauge field* $\mathbf{A}$ *satisfies the wave equation with a source term:*

$$-\Delta\mathbf{A} - \ddot{\mathbf{A}} = \frac{4\pi}{c}\mathbf{J} - \nabla\dot{\Phi} \tag{11.44}$$

Suppose $\nabla \cdot \mathbf{A} \neq 0$. Let $\Lambda$ be a solution of the Poisson's equation

$$\Delta\Lambda = \nabla \cdot \mathbf{A}$$

The gauge transformation

$$A'_\mu = A_\mu - \partial_\mu \Lambda$$

reproduces the the same field $F$ with $\mathbf{A}'$ satisfying the Coulomb gauge condition

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} - \Delta\Lambda = 0$$

$\Phi'$ It is determined by

$$\mathbf{E} = -\nabla\Phi' - \dot{\mathbf{A}}'$$

Taking the divergence of this we see that $\Phi'$ as a solution of Poisson's equation:

$$-\Delta\phi' = \nabla \cdot \mathbf{E} = 4\pi\rho \tag{11.45}$$

The wave equation for $\mathbf{A}'$ follows from Maxwell and the Coulomb gauge condition:

$$\begin{aligned}\frac{4\pi}{c}\mathbf{J} &= \nabla \times \mathbf{B} - \dot{\mathbf{E}} \\ &= \nabla \times (\nabla \times \mathbf{A}) + \nabla\dot{\mathbf{\Phi}} \\ &= -\Delta A + \nabla(\nabla \cdot \mathbf{A}) + \nabla\dot{\mathbf{\Phi}} \\ &= -\Delta A + \nabla\dot{\mathbf{\Phi}}\end{aligned} \tag{11.46}$$

**Remark 11.10** (Causality)**.** *The Coulomb gauge is a-causal: The scalar potential* $\phi$ *is fixed by the* instantaneous *charge distribution. You move a charge here and the scalar potential* $\phi$ *changes immediately everywhere. The fact that the scalar potential changes faster than light can not be used to transfer information faster than light because the fields are still causal, and only fields are measurable.*

**Remark 11.11** (Free space)**.** *When* $\rho = 0$ *we may take* $\phi = 0$ *together with* $\nabla \cdot \mathbf{A} = 0$.

## 11.8 Appendices

### 11.8.1 Cosmic rays: GZK limit

The cosmic microwave background (CMB) provides a shield that screens ultra high energy cosmic rays: A high energy charge proton can collide with a photon to produce a neutral pion converting much of the high kinetic energy of the proton to the pion mass. This makes the CMB a screen for high energy cosmic rays. The GZK limit says that protons with energies above $5 \times 10^{13}\ MeV$ are screened by the $3^\circ K$ thermal photons of the CMB.

Let us compute the threshold for particle (pion) production. The total energy-momentum of a proton with rapidity $\phi$ and counter-propagating photon in the plane is

$$p_\mu = m_P(\cosh\phi, \sinh\phi) + \hbar\omega(1, -1) = (p_0, p_1)$$

The energy in the center of mass frame, $E_{cm}$, is the scalar

$$E_{cm} = \sqrt{p_0^2 - p_1^1} = \sqrt{m_P^2 + 2m_P\hbar\omega e^\phi} = m_P + m_\pi$$

and the equality on the right expresses the fact that the two particles are at rest. This gives the threshold for pion production as

$$m_\pi^2 + 2m_P m_\pi = 2m_P\,\hbar\omega\, e^\phi$$

Since $m_\pi \ll m_P$ one finds a simple formula for the rapidity

$$e^\phi \approx \frac{m_\pi}{\hbar\omega}$$

The corresponding energy threshold is

$$m_p \cosh\phi \approx \tfrac{1}{2} m_p e^\phi \approx \frac{m_p m_\pi}{2 \times 3k_B} \approx 2.5 \times 10^{14}\ Mev$$

(This is factor 5 larger than the GZK estimate.)

**Exercise 11.12.** *Can you figure out why the estimate is too big?*

### 11.8.2 Laser cooling and optical molasses

Laser cooling is a cool application of the Doppler effect to slow down atoms. Think of the atom as a two level system with energy gap $E$. Suppose you point a laser beam with frequency $\hbar\omega < E$ at the atom. Atoms that move towards the light source will see bluer light and will be able to absorb the light if they move fast enough. This will slow the atom down. At the same time, slow atoms will be transparent to the light.

### 11.8.3 Covariant superposition

The wave equation is an algebraic equation in Fourier space. A solution can be written as

$$\phi(x) = \frac{1}{(2\pi)^2} \int d^4k \, \delta(k \cdot k) \, \tilde{\phi}(k) \, e^{ik \cdot x} \tag{11.47}$$

with $\tilde{\phi}(k)$ an arbitrary function of the 4-vector $k$. Of course, because of the $\delta$ function only the values that the function takes on the light-cone are relevant. This expression for a scalar wave $\phi$ is manifestly Lorentz invariant.

It is instructive to split the solution to the forward and backward light cone we have

$$\phi(k) = \theta(-t)\phi_<(\mathbf{k}) + \theta(t)\phi_>(\mathbf{k}) \tag{11.48}$$

We can carry out the time integration to get:

$$\phi_>(t, \mathbf{x}) = \frac{1}{(2\pi)^2} \int \frac{d\mathbf{k}}{2|\mathbf{k}|} \, \tilde{\phi}(|\mathbf{k}|, \mathbf{k}) \, e^{i(|\mathbf{k}|t - \mathbf{k} \cdot \mathbf{x})} \tag{11.49}$$

The forward light cone is associated with out going (retarded) waves. Similarly

$$\phi_<(t, \mathbf{x}) = \frac{1}{(2\pi)^2} \int \frac{d\mathbf{k}}{2|\mathbf{k}|} \, \tilde{\phi}(-|\mathbf{k}|, \mathbf{k}) \, e^{-i(|\mathbf{k}|t + \mathbf{k} \cdot \mathbf{x})} \tag{11.50}$$

the backward light-cone can be associated with incoming (advanced) waves. Note that in both cases, Lorentz invariance induces a weight on the three dimensional $\mathbf{k}$ space.

### 11.8.4 Monochromatic waves

Monochromatic waves are solutions of the wave equation whose time dependence is $e^{i\omega t}$. Hence

$$\Delta\phi = -{k_0}^2\phi \tag{11.51}$$

In Fourier space the solution is supported on the sphere

$$\mathbf{k} \cdot \mathbf{k} = k_0^2 \tag{11.52}$$

The smallest wave length that such a wave can accommodate is $2\pi/k_0$: The frequency limis the spatial resolution.

### 11.8.5 Evanescent waves

Near a planar boundary one can can arrange for monochromatic, evanescent plane wave solution to the wave equation: Plane waves in the half-space which decay in $x$ and propagate in the z-direction:

$$e^{-\kappa x} e^{ikz}, \quad k^2 = k_0^2 + \kappa^2 = \left(\frac{\omega}{c}\right)^2 + \kappa^2$$

The noteworthy fact about this waves is that the wave number $k$ in the z-direction can be much larger than the wave number associated with the frequency $\omega$ . Near the boundary $x = 0$ one finds waves with short wave lengths: $k \gg k_0 = \omega/c$.

**Example 11.13** (Transversal waves)**.** *The notion of transversality for evanescent waves is different from ordinary plane waves. For the wave*

$$\mathbf{E} = \mathbf{E}_0 e^{-\kappa x} e^{ikz}$$

*Gauss law* $\nabla \cdot \mathbf{E} = 0$ *reduces to*

$$\kappa(\mathbf{E}_0)_1 + ik(\mathbf{E}_0)_3 = 0$$

*which allows* $(E_0)_3 \neq 0$ *for a wave propagating the in z-direction.*

### 11.8.6 Waves in dielectric media: Birefringence:

In the absence of external sources, the time evolution of the fields in a dielectric is dictated by Faraday and Ampere laws

$$\dot{\mathbf{E}} + \nabla \times \mathbf{B} = 0, \quad \dot{\mathbf{H}} - \nabla \times \mathbf{D} = 0$$

subject to the constraints

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{B} = 0 \tag{11.53}$$

In Fourier space $(\mathbf{x}, t) \leftrightarrow (\omega, \mathbf{k})$ the differential equations reduce to algebraic equations

$$\omega\varepsilon^{-1}\,\mathbf{D} + \mathbf{k} \times \mu\mathbf{H} = 0, \quad \omega\mathbf{H} - \mathbf{k} \times \mathbf{D} = 0, \quad \mathbf{k} \cdot \mathbf{D} = 0, \quad \mathbf{k} \cdot \mu\mathbf{H} = 0$$

where $\mu$ and $\varepsilon$ are the constitutive relations. We assume that $\mu, \varepsilon$ are positive matrices. (Possibly functions of $\omega$.) Substitution gives for $\mathbf{D}$

$$\omega^2\varepsilon^{-1}\,\mathbf{D} + \mathbf{k} \times (\mu\mathbf{k} \times \mathbf{D}) = 0 \tag{11.54}$$

which can be written as a (generalized) eigenvalue problem for the $3 \times 3$ symmetric matrix

$$M_{jk}(\omega, \mathbf{k}) = \omega^2(\varepsilon^{-1})_{jk} + \varepsilon_{jmn}\varepsilon_{abk}\mu_{na}k_m k_b$$

Given $\mathbf{k}$ a non-trivial solution for the eigenvector $\mathbf{D}$ exists provided $\det M = 0$. This fixes the dispersion relation $\omega_j^2(\mathbf{k})$ with $j = 0, 1, 2$, the three eigenvalues of Eq. (11.54). One eigenvalue is always trivial since the matrix always 0 as an eigenvalue, corresponding to the eigenvector $\mathbf{k}$. There are, therefore, in general, only two non-trivial eigenvalues

**Exercise 11.14.** *Show that that if* $\varepsilon$ *and* $\mu$ *are symmetric, so is* $M$*.*

In the frame where $\varepsilon$ and $\mu$ are diagonal

$$M_{jk}(\omega,\mathbf{k}) = \omega^2(\varepsilon^{-1})_j\delta_{jk} + \mu_n\varepsilon_{njm}\varepsilon_{nbk}k_m k_b$$

In the special case that $\mu$ is a scalar

$$M_{jk}(\omega,\mathbf{k}) = \big(\omega^2(\varepsilon^{-1})_j - \mu\mathbf{k}\cdot\mathbf{k}\big)\delta_{jk} + \mu k_j k_k$$

For $\mathbf{k}$ in the principal direction $j$ we get

$$\omega^2 = \varepsilon_j\mu\, k^2$$

The wave propagates at different speeds $\sqrt{e_j\mu}$ along the principal directions of $\varepsilon$. This is birefringence.

### 11.8.7 3D glasses

When you view a 3D movie, the 3D glasses transmit a picture with right circular polarization to, say, the right eye and left circular polarization to the left eye.

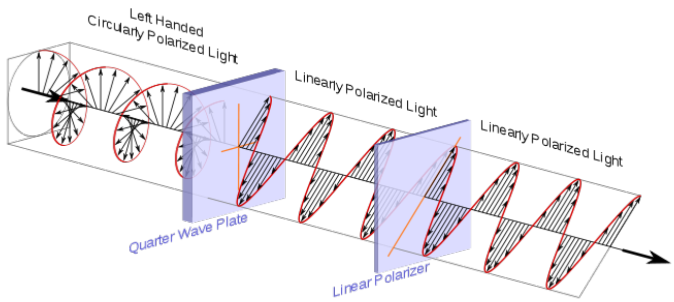



Figure 11.8: An arrangement that transmits right circular polarization.

**Exercise 11.15.** *Can you give an (ergonomic) argument why spectators would prefer circular to linear polarization?*

**Exercise 11.16.** *Define quarter wave plate as the rotation of the Poincare sphere that turns circular polarization to linear. Show that it is represented by Hadamard gate H*

$$\sqrt{2}H = \sigma_3 + \sigma_1$$

**Exercise 11.17.** *Explain why the filtering associated by linear polarizers can be described by the projections*

$$P_{H,V} = \frac{\mathbb{1} \pm \sigma_1}{2}$$

It follows from the two exercises above that the right and left glasses can be represented by $2\times 2$ matrices

$$g_1 = P_H H, \quad g_2 = P_V H$$

**Exercise 11.18.** *Give a physical interpretation of the identity*

$$P_V\sigma_3 = \sigma_3 P_H$$

*in terms of rotation of the glasses.*

**Exercise 11.19.** *Explain why holding the glasses backwards is represented by transposition:*

$$g_j \Longleftrightarrow g_j^t$$

If you place glass 1 rotated by $\pi/2$ behind glass 2 inverted the joint system is represented by represented by the matrix product $\sigma_3 g_1\, g_2^t$. A computation gives 0 which means that no light passes through.

**Exercise 11.20.** *Show that there are 64 ways of arranging the pair of glasses. How many of these let no light through.*

**Bibliography**

F. John, Partial differential equations,

# Chapter 12

# Radiation

## 12.1 Wave equation with arbitrary source term

The retarded Green function for the wave equation in 3+1 dimensions is

$$G_{>}(x) = \frac{1}{2\pi}\,\theta(x^0)\delta\big(s\big) = \frac{\delta(ct-r)}{4\pi r}, \quad s = x\cdot x, \quad r = |\mathbf{x}| \geq 0 \tag{12.1}$$

The $\theta$ function guarantees causality: The past influences the present. The delta function says that in 3+1 dimensions signals propagate on the light-cone. Huygens principle (in the strong form) holds.

By the homogeneity of Minkowski space, for source term located at the space-time point $y$

$$\Box G_{>}(x-y) = \delta(x-y)$$

By the linearity of the wave equation the retarded solution of the wave equation, $\phi$, generated by an arbitrary[1] source $\rho$

$$\Box\phi = \rho(x) \tag{12.2}$$

is

$$\phi_\rho(x) = \int d^4y\, G_{>}(x-y)\rho(y) \tag{12.3}$$

Solving the scalar wave equation is reduced to computing an integral.

### 12.1.1 Scalar wave generated by a moving point source

As a preparation for studying the radiation of electromagnetic waves, consider the simpler problem of radiation of scalar waves generated by a point source

[1]Some condition on the localization of the sources should be imposed. This is related to Olber's paradox: If you assume constant density of stars, and that intensity of radiation falls like $r^{-2}$ the night sky shoudl be as bright as the sun.

moving on a world line $z = \big(ct, \mathbf{z}(t)\big),\ -\infty < t < \infty$. The motion is assumed to be that of a real particle so the 4-velocity is time-like. We take the source to be

$$\rho(\mathbf{x}, t) = \frac{4\pi}{c}\delta^{(3)}\big(\mathbf{x} - \mathbf{z}(t)\big) \tag{12.4}$$

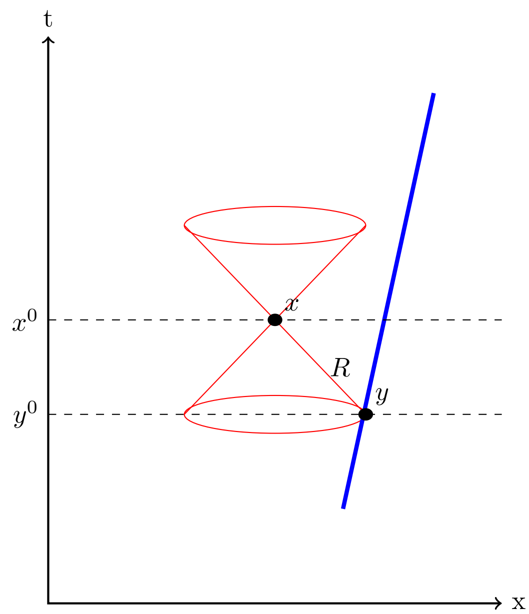



Figure 12.1: The blue line is the world line of the point source. The wave is observed at the black dot $x$. The backward light cone from $x$ intersects the (blue) world line at the black dot $y$. Since the velocity is time like the point of intersection is unique. $R$ is the 4 vector $x - z(y^0)$.

The retarded wave is:

$$\begin{aligned} c\phi_\delta(x) &= 2\int d^4y\ \underbrace{\delta^{(3)}\big(\mathbf{y} - \mathbf{z}(y^0)\big)}_{source}\ \underbrace{\delta\big((x-y)\cdot(x-y)\big)\,\theta(x^0 - y^0)}_{Green} \\ &= 2\int dy^0\ \delta\big(R\cdot R\big)\theta(x^0 - y^0), \quad R = (x^0 - y^0, \mathbf{x} - \mathbf{z}(y^0)) \end{aligned}$$

Since the orbit $\mathbf{z}(z^0)$ is time-like, a single point contributes: a single particle has a single image[2]. To compute the remaining time-integral use

$$\int \delta\big(s(y)\big)dy = \frac{1}{|s'(y_0)|}, \quad s = R\cdot R, \quad s(y_0) = 0. \tag{12.5}$$

[2]Mirrors and lenses can create multiple images, of course.

Clearly, the time derivative of $s$ is related to the velocity of the source, and it is natural to express it in terms of the 4-velocity

$$\begin{aligned}\frac{ds}{dy^0} &= 2R_\mu \frac{dR^\mu}{dy^0} = -2R_\mu \frac{dz^\mu}{dy^0} \\ &= -2R_\mu \frac{dz^\mu}{d\tau}\frac{d\tau}{dy^0} = -2\,\frac{R_\mu u^\mu}{c\gamma} \\ &= -2\,\frac{R\cdot u}{c\gamma}\end{aligned}$$

The wave $\phi(x)$ at the observing point $x$ is therefore given by the deceptively simple formula

$$\phi(x) = \frac{\gamma(y^0)}{|R\cdot u(y^0)|}, \quad R = x - y, \quad R\cdot R = 0 \tag{12.6}$$

which is manifestly causal and satisfies Huygens principle.

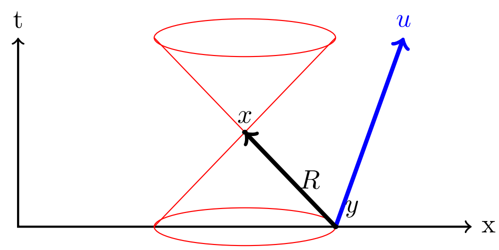



Figure 12.2: The blue 4-velocity is time-like and the 4-vector R is light like. Their Minklowsky scalar product is negative and can be close to zero.

The formula is simple but *implicit*. The right hand side *is not* an explicit function of the argument $x$: You need to know $\gamma(y^0)$, $u(y^0)$ and $R = x - y$ and in particular, the *earlier time* $y^0$ (see Fig. 12.1.1). This time is determined as the solution of the equation

$$(x^0 - y^0)^2 = \big(\mathbf{x} - \mathbf{z}(y^0)\big)^2$$

which may be arbitrarily complicated if the orbit $\mathbf{z}(z^0)$ is complicated.

The amplitude of decays like $1/R$, which is characteristic of waves.

## 12.2 Maxwell equation in the Lorenz gauge

The inhomogeneous Maxwell equations are

$$\partial^\mu F_{\mu\nu} = -\frac{4\pi}{c}\, j_\nu \tag{12.7}$$

Expressed in terms of the potentials, (this guarantees the homogeneous equations), one gets a system of second order PDE

$$\partial^\mu{}_\mu A_\nu - \partial^\mu{}_\nu A_\mu = -\frac{4\pi}{c}\, j_\nu \tag{12.8}$$

In the Lorenz gauge, $\partial^\mu A_\mu = 0$, Maxwell equations reduce to 4 *decoupled* wave equations

$$\Box A_\nu = \partial^\mu{}_\mu A_\nu = -\frac{4\pi}{c}\, j_\nu \tag{12.9}$$

The equations are coupled through the Lorenz gauge condition. If the current $j^\mu$ is not conserved then the derivation is inconsistent with the Lorenz gauge. Conversely, if current is conserved, then Lorenz gauge condition follows for all times provided the initial data for $A_\mu$ and $\dot{A}_\mu$ satisfy it.

**Exercise 12.1.** *Show that if one imposes the Lorentz gague condition as initial data for the wave equation, then the Lorenz gauge condition holds for all times provided current is conserved.*

## 12.3 Lienard-Wiechert: Retarded potentials

The Maxwell equations, Eqs. (12.9), can be viewed as 4 independent copies of the scalar wave equation with given source terms. We can therefore use the solution from the previous section to the source of Eq. (12.9) where

$$j_\nu(y) = e\frac{4\pi}{c}\delta^{(3)}\big(\mathbf{y}-\mathbf{z}(y^0)\big)\underbrace{v_\nu(y^0)}_{\text{lab velocity}} = e\frac{4\pi}{c}\frac{\delta^{(3)}\big(\mathbf{y}-\mathbf{z}(y^0)\big)}{\gamma}\underbrace{u_\nu(y^0)}_{4-velocity} \tag{12.10}$$

Comparing with Eqs. (12.4,12.6) for scalar waves we see that conveniently $\gamma$ disappears, and the retarded potentials are:

$$A_\nu(x) = e\frac{u_\nu}{|R\cdot u|} = -e\frac{u_\nu(y^0)}{R\cdot u(y^0)} \tag{12.11}$$

The absolute value was removed by taking into account that $R$ is forward *light-like* and $u$ forward *time-like* so $R\cdot u < 0$.

The result admits the following (a-posteriori) interpretation: The vector potential, being a 4-vector, must be of the form

$$(scalar)(vector)_\mu$$

We have (at least) two 4-vectors at our disposal: $R$ and $u$. Between these we can form 3 scalars: One interesting $R\cdot u$ and two uninteresting $u\cdot u = -c^2$ and $R\cdot R = 0$. This, plus dimension analysis and the limit case of a charge at rest determines Eq. (12.11).

The result can also be viewed as the covariant form of Coulomb law:

$$A_\nu(x) = \frac{e}{|\mathbf{x}|}(1,0,0,0) = -\frac{e}{c|\mathbf{x}|}\underbrace{(-c,0,0,0)}_{u_\nu} \Longrightarrow -e\,\frac{u_\nu}{R\cdot u}$$

### 12.3.1 The Lorenz Gauge condition

We still need to verify the Lorenz gauge condition.

*A clever argument*: Since the condition is Lorentz invariant, it is sufficient to verify it in some Lorentz frame. So let us do that in the frame where the charge is instantaneously at rest at the origin (at the early time). Then,by Eq. 12.11

$$\partial^\mu A_\mu = \partial^0 A_0, \quad A_0 = -\frac{e}{|\mathbf{x}|}$$

when you change $t$ of the event $x$ the distance $\mathbf{x}$ does not change because the particle is at rest. Hence

$$\partial^0 A_0 = 0$$

*An honest Computation*: The reason for doing an honest computation is that this will force us to derive the identity that describes how the retarded time depends upon variation of the observation event $x$ which we shall need when deriving a formula for the fields. The identity is:

$$\partial_\mu \tau = \frac{R_\mu}{R \cdot u} \tag{12.12}$$

It follows by differentiating the light-like condition relating the events: $R \cdot R = 0$

$$0 = \tfrac{1}{2}\partial_\mu (R \cdot R) = R_\alpha\, \partial_\mu (x^\alpha - z^\alpha) = R_\mu - R \cdot u\, (\partial_\mu \tau)$$

Back to the honest verification of the Lorenz gauge condition:

$$0 = \partial_\mu A^\mu = -e\partial_\mu \left(\frac{u^\mu}{R \cdot u}\right)$$

This will hold provided

$$(R \cdot u)^2 \partial_\mu \left(\frac{u^\mu}{R \cdot u}\right) = (R \cdot u)\, \partial_\mu u^\mu - u^\mu \partial_\mu (R \cdot u) \overbrace{=0}^{?} \tag{12.13}$$

To verify that this is indeed so, let us prepare

$$\partial_\mu u^\alpha = \dot{u}^\alpha (\partial_\mu \tau), \quad \partial_\mu R^\alpha = \delta^\alpha_\mu - u^\alpha (\partial_\mu \tau)$$

Substituting this in Eq. (12.13) and using Eq. (12.12) we find

$$\begin{aligned}(R \cdot u)\, \partial_\mu u^\mu - u^\mu \partial_\mu (R \cdot u) &= (R \cdot u)\, \dot{u}^\mu (\partial_\mu \tau) - u^\mu u_\alpha \partial_\mu R^\alpha - u^\mu R_\alpha \dot{u}^\alpha (\partial_\mu \tau) \\ &= (R \cdot u)\, \frac{\dot{u} \cdot R}{R \cdot u} - u^\mu u_\alpha \partial_\mu R^\alpha - u^\mu R_\alpha \dot{u}^\alpha \frac{R_\mu}{R \cdot u} \\ &= -u^\mu u_\alpha \partial_\mu R^\alpha \\ &= -u \cdot u + u^\mu \partial_\mu \tau \\ &= -u \cdot u + u \cdot u = 0\end{aligned}$$

We have verified that the solution, Eq. 12.11, indeed satisfies the Lorenz condition.

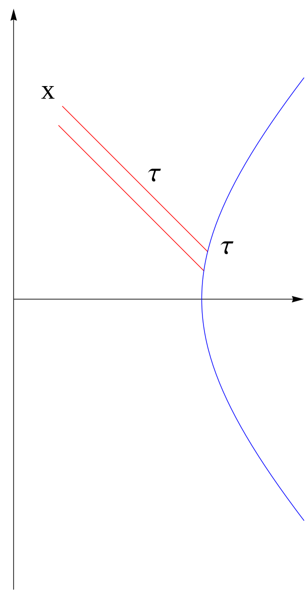



Figure 12.3: The self time $\tau$ parametrizes the blue orbit. It can be extended to a function on space time by pushing the value of $\tau$ to the forward light-cone. The figure illustrates how $\tau$ changes when the point of observation $x$ changes. The red lines are light-like.

## 12.4 Lienard Wiechert formula for retarded field

To find the fields we need to differentiate the potentials with respect to the space-time coordinates $x^\mu$. Formally,

$$F_{\mu\nu} = \boldsymbol{\partial}_\mu A_\nu - \boldsymbol{\partial}_\nu A_\mu = \boldsymbol{\partial}_{[\mu} A_{\nu]} \tag{12.14}$$

and the right hand side is just a convenient notation. The word formal above refers to the fact that in taking the partial derivatives we need to remember that $\tau$, the retarded time, is a function of the point of observation $x$, fig 12.3. If we treat $\partial_x$ independent of $\tau$ then $\boldsymbol{\partial}_\mu$ of Eq. 12.14 needs to be interpreted as

$$\boldsymbol{\partial}_\mu = \frac{\partial}{\partial x^\mu} + \left(\frac{\partial \tau}{\partial x^\mu}\right)\frac{\partial}{\partial \tau} = \frac{\partial}{\partial x^\mu} + \left(\frac{R_\mu}{R\cdot u}\right)\frac{\partial}{\partial \tau} \tag{12.15}$$

The $x^\mu$ differentiation cares about the location of the observer, while the $\tau$ differentiation cares about the location of the charge.

Using the explicit form of the potentials of the previous section:

$$-\boldsymbol{\partial}_\mu A_\nu = e\,\boldsymbol{\partial}_\mu\left(\frac{u_\nu}{R\cdot u}\right) = -e\,\frac{u_\mu u_\nu}{(R\cdot u)^2} + e\left(\frac{R_\mu}{R\cdot u}\right)\partial_\tau\left(\frac{u_\nu}{R\cdot u}\right) \tag{12.16}$$

Because $F$ is anti-symmetric, the first term in Eq. (12.16) drops upon anti-symmetrization and only the second term contributes

The field $F$ depend on the location, velocity $u$ and the acceleration $\dot{u}$ of the charge at the early time. It does not depend on any higher derivatives, e.g. the jerk $\ddot{u}$. Now compute:

$$\begin{aligned} \partial_\tau \left(\frac{u_\nu}{R\cdot u}\right) &= \frac{\dot{u}_\nu}{R\cdot u} - \frac{u_\nu}{(R\cdot u)^2}\,\partial_\tau(R\cdot u) \\ &= \frac{\dot{u}_\nu}{R\cdot u} + \frac{u_\nu}{(R\cdot u)^2}\,u\cdot u - \frac{u_\nu}{(R\cdot u)^2}\,R\cdot\dot{u} \\ &= \frac{\dot{u}_\nu}{R\cdot u} - c^2\,\frac{u_\nu}{(R\cdot u)^2} - \frac{u_\nu}{(R\cdot u)^2}\,R\cdot\dot{u} \end{aligned} \tag{12.17}$$

Consequently

$$\left(\frac{R_\mu}{R\cdot u}\right)\partial_\tau\left(\frac{u_\nu}{R\cdot u}\right) = \frac{R_\mu \dot{u}_\nu}{(R\cdot u)^2} - c^2\,\frac{R_\mu u_\nu}{(R\cdot u)^3} - \frac{R_\mu u_\nu}{(R\cdot u)^3}\,R\cdot\dot{u} \tag{12.18}$$

We get $F$ by anti-symmetrizing:

$$F_{\mu\nu} = -e\,\underbrace{\left(\frac{R_{[\mu}\dot{u}_{\nu]}}{(R\cdot u)^2} - \frac{R_{[\mu}u_{\nu]}}{(R\cdot u)^3}\,R\cdot\dot{u}\right)}_{radiation} + e\,\underbrace{c^2\,\frac{R_{[\mu}u_{\nu]}}{(R\cdot u)^3}}_{\text{"}Coulomb\text{"}} \tag{12.19}$$

Since $u = O(c)$, the last term, is order $O(c^0)$. It decays with distance like $R^{-2}$. This is, essentially, the Coulomb term. The first two terms are proportional to the acceleration and their velocity dependence is of order $O(c^{-2})$. They decay like $R^{-1}$. These are the radiating terms.

### 12.4.1 Interpretation

The Lienard-Wiechert formula is complicated and at first also opaque. It may be useful to view it from general principles.

We have three vectors in the problem:

- $R$, the light-like vector connecting the point of observation and the source.
- $u$ the particle 4-velocity.
- $\dot{u}$ the 4-acceleration.

From these we can make three interesting scalars

$$R\cdot u, \quad R\cdot\dot{u}, \quad \cancel{\dot{u}\cdot\dot{u}} \tag{12.20}$$

The remaining scalars are less interesting

$$R\cdot R = 0, \quad u\cdot u = -c^2, \quad u\cdot\dot{u} = 0 \tag{12.21}$$

To see why $\dot{u}\cdot\dot{u}$ has been cancelled, observe that

- $F$ must be (at most) linear in the acceleration.

This is because the potential did not depend on the acceleration at all. As a consequence, the scalar $\dot{u} \cdot \dot{u}$ should not appear and the scalar $R \cdot \dot{u}$ can only appear in the numerator.

We can now reconstruct all the three terms in $F$ just by the fact that $F$ is a tensor and dimension analysis: From the tensorial properties of $F$ it must be of the form

$$(tensor)_{\mu\nu} = (scalar)\,(vector)_\mu (vector)_\nu$$

Since $F$ has dimension of $[charge][length^{-2}]$ and $u$ has the dimension of $c$, one possible term is

$$ec^2 \frac{R_{[\mu}\, u_{\nu]}}{(R \cdot u)^3}$$

which is the last term in Eq. (12.19). You can even get the numerical factor (and the sign) by looking at the limiting case of the Coulomb field of a particle at rest where $u_\mu = (-c, 0, 0, 0)$.

One possible term proportional to the acceleration is $\dot{u}$ is

$$e \frac{R_{[\mu}\, \dot{u}_{\nu]}}{(R \cdot u)^2}$$

which gives the first term up tod numerical factor. The middle term is obtained similarly.

## 12.5 Accelerating particle in its rest frame

The formula for $F$ simplifies for a particle instantaneously at rest at the origin (at the early time), i.e.

$$y^0 = 0, \quad \mathbf{z}(y^0) = 0, \quad u^\mu = (c, 0, 0, 0) \quad R \cdot u = -c|\mathbf{x}|. \tag{12.22}$$

We can always achieve this by choosing appropriate Lorentz frame. This determines $F$ on the forward light-cone $(|\mathbf{x}|, \mathbf{x})$. Lets us examine $\mathbf{B}$ first and then $\mathbf{E}$.

### 12.5.1 The Magnetic field:

Since the spatial components of $u$ vanish, we have (in Cartesian coordinates)

$$\begin{aligned} F_{ij}(|\mathbf{x}|, \mathbf{x}) &= \varepsilon_{ijk} B^k(|\mathbf{x}|, \mathbf{x}) \\ &= -e \left( \frac{R_{[i}\dot{u}_{j]}}{(R \cdot u)^2} - \frac{\overbrace{R_{[i}u_{j]}}^{=0}}{(R \cdot u)^3}\, R \cdot \dot{u} \right) + ec^2 \frac{\overbrace{R_{[i}u_{j]}}^{=0}}{(R \cdot u)^3} \\ &= -\frac{e}{c^2} \frac{R_{[i}\dot{u}_{j]}}{|\mathbf{x}|^2} \\ &= \frac{e}{c^2} \frac{(\mathbf{a}(0) \times \mathbf{x})_k}{|\mathbf{x}|^2} \end{aligned} \tag{12.23}$$

where **a**, the 3-vector of acceleration $\dot{u}_\mu = (0, \mathbf{a})$, is orthogonal to $u = (c, 0)$.

It follows that, 3 vector of magnetic field on the light-cone emanating from the origin, $(|\mathbf{x}|, \mathbf{x})$ is

$$\mathbf{B}(|\mathbf{x}|, \mathbf{x}) = \frac{e}{c^2} \frac{\mathbf{a}(0) \times \mathbf{x}}{|\mathbf{x}|^2} \tag{12.24}$$

The main conclusions we draw from this are

1. The field decays like the inverse distance from the source.

2. The field is perpendicular to both the line of sight and the acceleration vector.

### 12.5.2 The electric field

Recall that $F_{0j} = -E_j$. For a particle at rest at the origin at lab time $y^0 = 0$ $R \cdot u = -c|\mathbf{x}|$ and $R_0 = -|\mathbf{x}|$. Hence, on the light-cone emanating from the origin

$$\begin{aligned} E_j = -F_{0j} &= e \left( \frac{R_{[0}\dot{u}_{j]}}{(R \cdot u)^2} - \frac{R_{[0}u_{j]}}{(R \cdot u)^3} R \cdot \dot{u} \right) - ec^2 \frac{R_{[0}u_{j]}}{(R \cdot u)^3} \\ &= \frac{e}{c^2|\mathbf{x}|^3} \left( |\mathbf{x}|\, R_{[0}\dot{u}_{j]} - \frac{1}{c} R_{[0}u_{j]}\, R \cdot \dot{u} \right) + \frac{e}{c|\mathbf{x}|^3} R_{[0}u_{j]} \\ &= \frac{e}{c^2|\mathbf{x}|^3} \left( -(\mathbf{x} \cdot \mathbf{x})\, \mathbf{a}_j + (\mathbf{x} \cdot \mathbf{a})\, \mathbf{x}_j \right) + \frac{e}{r^2} \mathbf{x}_j \end{aligned}$$

Since

$$(\mathbf{x} \cdot \mathbf{x})\mathbf{a} - (\mathbf{x} \cdot \mathbf{a})\, \mathbf{x} = -\big(\mathbf{x} \times (\mathbf{x} \times \mathbf{a})\big)$$

We can collect the above to a vector identity

$$\begin{aligned} \mathbf{E}(|\mathbf{x}|, \mathbf{x}) &= \frac{e}{c^2|\mathbf{x}|} \big(\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \mathbf{a})\big) + \frac{e}{|\mathbf{x}|^2} \hat{\mathbf{x}} \\ &= \mathbf{B}(|\mathbf{x}|, \mathbf{x}) \times \hat{\mathbf{x}} + \frac{e}{|\mathbf{x}|^2} \hat{\mathbf{x}} \end{aligned} \tag{12.25}$$

The longitudinal part is Coulomb and the transversal part is radiation. **E** and **B** are mutually orthogonal. The two Lorentz scalars are

$$\mathbf{E} \cdot \mathbf{B} = 0, \quad \mathbf{E}^2 - \mathbf{B}^2 = -\frac{e^2}{|\mathbf{x}|^4}$$

**Exercise 12.2.** *Show that the Poynting vector is*

$$\frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{e^2}{4\pi c^3 |\mathbf{x}|^2}\, \hat{\mathbf{x}}$$

### 12.5.3 Magnetic field in the far field region

In the case that the charge is not quite at the origin and not quite stationary, one can still write a not too ugly expression for $B$ in the important case of the far field where $|\mathbf{x}|$ is the largest length scale in the problem. We now retain only the terms that decay like $1/R$

$$\begin{aligned} F_{ij}(x) &= \varepsilon_{ijk} B^k(|\mathbf{x}|, \mathbf{x}) \\ &\approx -e\left(\frac{R_{[i}\dot{u}_{j]}}{(R\cdot u)^2} - \frac{R_{[i}u_{j]}}{(R\cdot u)^3}\, R\cdot\dot{u}\right) \end{aligned} \tag{12.26}$$

In the far field $|\mathbf{z}|$ is small compared to $|\mathbf{x}|$ and can approximate

$$R = (x^0 - y^0, \mathbf{x} - \mathbf{z}(y^0)) \approx (|\mathbf{x}|, \mathbf{x}) \tag{12.27}$$

(We have used the fact that $R$ is light-like.) If, in addition, the charge is non-relativistic

$$u_\mu \approx (c, 0) \tag{12.28}$$

and we obtain

$$\varepsilon_{ijk} B^k(x) \approx -\frac{e}{c^2}\frac{R_{[i}\dot{u}_{j]}}{r^2} \approx -\frac{e}{c^2}\frac{x_i a_j - x_j a_i}{r^2}, \quad r = |\mathbf{x}| \tag{12.29}$$

We have essentially recovered the results Eq. 12.24 in the far-field also for particles that need not be stationary and at the origin

$$\mathbf{B}(x) \approx \frac{e}{c^2}\frac{\mathbf{a}(y^0)\times\mathbf{x}}{|\mathbf{x}|^2} \tag{12.30}$$

Note that the result now implicitly depends on the retarded time $y^0$.

## 12.6 Retardation from a distant source

To compute the $F_{\mu\nu}$ from Lienard-Wiechert formula we need to know

$$(R, \quad u, \quad \dot{u}) \tag{12.31}$$

at the time time $y^0$ when the radiation has been emitted. This time is determined as the solution of the equation

$$x^0 - y^0 = |\mathbf{x} - \mathbf{z}(y^0)|$$

where $\mathbf{z}(y^0)$, the obit of the source (charge), is given. This is an implicit equation for $y^0$. A geometric solution is given in Fig. 12.1.1. In general, one can not hope to find and analytic solution: A-priori $\mathbf{z}(y^0)$ could be an arbitrarily complicated function that can not be inverted explicitly. The best we can hope for is to find approximate solutions in simple cases when there is a small or a large parameter that we can use in making successive approximations.

Let us address the accurately we need to estimate the radiation time $y^0$ to estimate $R$. Recall that

$$R = (x - y^0, \mathbf{x} - \mathbf{z}(y^0)) \tag{12.32}$$

is light-like. So, it is sufficient to have a good estimate of its spacial part. Now, if the source is distant from the point of observation, and if it is confined to a relatively small ball in space,

$$|\mathbf{z}| \leq \ell \ll |\mathbf{x}| \tag{12.33}$$

When $|\mathbf{x}|$ is large, we need to approximate $|\mathbf{x} - \mathbf{z}|$ not just to order $|\mathbf{x}|$, but also to order $|\mathbf{x}|^0 = O(1)$, but we may neglect $|\mathbf{x}|^{-1}$

$$|\mathbf{x} - \mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{z}|^2 - 2\mathbf{x} \cdot \mathbf{z} = |\mathbf{x}|^2 \left( 1 - 2\frac{\mathbf{x} \cdot \mathbf{z}}{|\mathbf{x}|^2} + O\left(\frac{\ell}{|\mathbf{x}|}\right)^2 \right) \tag{12.34}$$

Approximating $\sqrt{1 + 2\varepsilon} \approx 1 + \varepsilon$ gives

$$|\mathbf{x} - \mathbf{z}| \approx |\mathbf{x}| \left( 1 - \frac{\mathbf{x} \cdot \mathbf{z}}{|\mathbf{x}|^2} \right) \tag{12.35}$$

The implicit equation for the retardation $y^0$ for a distant source simplifies to:

$$x^0 - y^0 \approx |\mathbf{x}| - \hat{\mathbf{x}} \cdot \mathbf{z}(y^0) \tag{12.36}$$

Let us rewrite the equation in the form

$$y^0 \approx x^0 - |\mathbf{x}| + \hat{\mathbf{x}} \cdot \mathbf{z}(y^0) \tag{12.37}$$

This is still an implicit equation for $y^0$, so we shall need to make some further approximations if we want to find explicit approximate expression for $y^0$.

### 12.6.1 The dipole approximation

We now need to return to the question how accurately we need to solve Eq. 12.37 to get sufficiently good estimates of $u(y^0)$ and $\dot{u}(y^0)$ and equivalently, $\dot{\mathbf{z}}(y^0)$ and $\ddot{\mathbf{z}}(y^0)$. If the source has characteristic frequency $\omega$, then we need to have an estimate of the time at the source with accuracy better than

$$\omega\, \delta t \ll 1 \Longleftrightarrow \omega\, \delta y^0 \ll c \tag{12.38}$$

For example, in the case that the source is harmonic, this means that we need to know the phase of the source $y^0\omega/c$ to an accuracy that is much better than $2\pi$. We increasingly better accuracy as $\omega$ gets large.

### 12.6.2 Dipole approximation: Successive approximations

We can solve the equation 12.37 by iteration. Assume that for all times $|\mathbf{z}(x^0| \leq \ell$ is near the origin, and so small compared to $|\mathbf{x}|$. This motivates starting the iteration with

$$\begin{aligned} y_1^0 &= x^0 - |\mathbf{x}| \\ y_2^0 &= x^0 - |\mathbf{x}| + \hat{\mathbf{x}} \cdot \mathbf{z}(y_1^0) = x^0 - |\mathbf{x}| + \hat{\mathbf{x}} \cdot \mathbf{z}(x^0 - |\mathbf{x}|) \\ \ldots \end{aligned} \tag{12.39}$$

Let us see how good the successive approximations are. We can estimate the error in $y_1^0$ by comparing with the defining equation for $y^0$:

$$y^0 - y_1^0 = \hat{\mathbf{x}} \cdot \mathbf{z}(y^0) = O(\ell) \tag{12.40}$$

This is a dimension-full error and so has no natural notion of size, it is neither small not large. Eq. 12.38 allows us to translated the error to a dimension-less quantity by multiplying by $\omega/c$

$$\frac{\omega\ell}{c} = \frac{2\pi\ell}{\lambda} \tag{12.41}$$

We can phrase this as the statement that if the wavelength of the radiation $\lambda$ is much larger than the source, $y_1^0$ is a good approximation. This is the dipole approximation.

$y_2^0$ is an even better approximations. Substituting $y_2^0$ in Eq. 12.37 gives the error

$$\begin{aligned} y^0 - y_2^0 &= \hat{\mathbf{x}} \cdot \mathbf{z}(y^0) - \hat{\mathbf{x}} \cdot \mathbf{z}(x^0 - |\mathbf{x}|) \\ &\approx -\hat{\mathbf{x}} \cdot \dot{\mathbf{z}}(y^0)(y^0 - x^0 + |\mathbf{x}|) \\ &= -\hat{\mathbf{x}} \cdot \dot{\mathbf{z}}(y^0)\hat{\mathbf{x}} \cdot \dot{\mathbf{z}}(y^0) \\ &= O(\omega\ell^2/c) \end{aligned}$$

This translates to the dimensionless error

$$\left(\frac{\omega\ell}{c}\right)^2 = \left(\frac{2\pi\ell}{\lambda}\right)^2 \tag{12.42}$$

This is a quadratic improvement relative to the error in $y_1^0$. $y_2^0$ gives an explicit equation for $y^0$ given $x$, the point of observation

$$y_2^0 \approx x^0 - |\mathbf{x}| + \hat{\mathbf{x}} \cdot \mathbf{z}\left(x^0 - |\mathbf{x}|\right) \tag{12.43}$$

This completes our discussion of the dipole approximation. The approximation is applies to atomic systems radiating visible light: The size of atomic systems is $O(1\,[\text{Å}])$ while visible light has wavelength $O(5000\,[\text{Å}])$. The approximation does not hold, of course, for X-rays.

In the case of your cell phone $\omega\ell/c = O(1)$ and one needs to go beyond the dipole approximation.

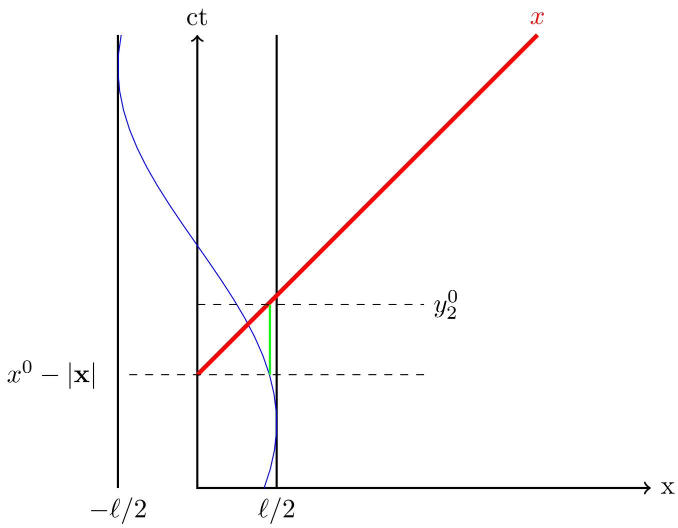



Figure 12.4: The blue line gives the orbit of the charge. The lowest oredr dipole approximation is $x^0 - |\mathbf{x}|$. A better approximation is $y_2^0$ which takes into account the position of the charge at $x^0 - |\mathbf{x}|$. One could get better approximations by also taking into account the velocity of the charge.

### 12.6.3 Radiation from a charge in Harmonic motion

As an application consider a charge undergoing non-relativistic harmonic motion with

$$\mathbf{z}(y^0) = \mathbf{z}_0 \, e^{i\omega\tau}, \quad y^0 = c\tau \tag{12.44}$$

$\mathbf{z}_0$ is a vector with complex amplitudes and taking the real part is implicit in the formula. Thus, for example $\mathbf{z}_0 = \ell(1, i, 0)/\sqrt{2}$ describes circular motion with radius $\ell$ in the plane. The dipole approximation gives for the retarded time $\tau$

$$\begin{aligned} c\tau &= y^0 \\ &\approx x^0 - r + \hat{\mathbf{x}} \cdot \mathbf{z}\left(x^0 - r\right) \\ &= x^0 - r + \hat{\mathbf{x}} \cdot \mathbf{z}_0 e^{i\omega(x^0 - r)}, \quad r = |\mathbf{x}| \end{aligned} \tag{12.45}$$

By Eq. 12.30

$$\begin{aligned} \mathbf{B}(x) &\approx \frac{e}{c^2} \frac{\mathbf{a}(y^0) \times \hat{\mathbf{x}}}{r} \\ &= -\frac{e\omega^2}{c^2} \mathbf{z_0} \times \hat{\mathbf{x}} \frac{e^{i\omega\tau}}{r} \end{aligned} \tag{12.46}$$

The field is proportional to s spherical wave

$$\frac{e^{ik(x^0 - r)}}{r} \tag{12.47}$$

The most interesting thing to observe is that this outgoing spherical wave is a consequence of retardation.

**Exercise 12.3.** *Consider the radiation from two opposite charges $\pm e$ executing harmonic motion with opposite amplitudes $\pm\mathbf{z}_0$. In this case one needs to consider the retardation beyond the leading order in the dipole approximation, i.e. one needs to keep the last term in Eq. 12.45.*

### 12.6.4 Many particles

The radiation fields from many particles with prescribed orbits is, by linearity of the Maxwell equation, the sum of the radiation of the individuals ones. In general, each particle will have its own retarded time, and the formulas remain implicit. A simplification occurs in the dipole approximation for " small antenna" where all the charges share the same retardation.

The dipole moment of a large collection of charges is:

$$\mathbf{d}(t) = \sum e_j \mathbf{z}_j\,(t) \tag{12.48}$$

and we assume that all the orbits $\mathbf{z}_j$ are such that the dipole approximation applies. In this case all the charges have the same retardation and the magnetic field is simple

$$\mathbf{B}(\mathbf{r},t) = \frac{\ddot{\mathbf{d}}(t-r/c)\times\hat{\mathbf{x}}}{c^2|\mathbf{x}|} \tag{12.49}$$

,

## 12.7 Power

The power emitted by dipole can be computed from Poynting

$$\mathbf{P} = c\frac{\mathbf{E}\times\mathbf{B}}{4\pi} \tag{12.50}$$

Both **E** and **B** lie in the plane perpendicular to the line of sight so $P$ is parallel to $\hat{r}$.

*Linear Dipole:* Suppose that $\mathbf{a} = a\hat{\mathbf{z}}$. The magnitude of $P$ in the direction of the spherical angle $\theta$ relative to the z-axis is

$$P(\theta) = c\frac{E^2}{4\pi} = \frac{e^2a^2}{4\pi c^3}\,\frac{\sin^2\theta}{r^2} \tag{12.51}$$

The power through a spherical shell of radius $r$ is then

$$P_T = 2\pi r^2 \int d\theta\,\sin\theta\,P(\theta) \tag{12.52}$$

Evidently

$$\int d\theta\,\sin\theta\,\sin^2\theta = -\int d(\cos\theta)\,(1-\cos^2\theta) = 2\left(1-\tfrac{1}{3}\right) = \frac{4}{3} \tag{12.53}$$

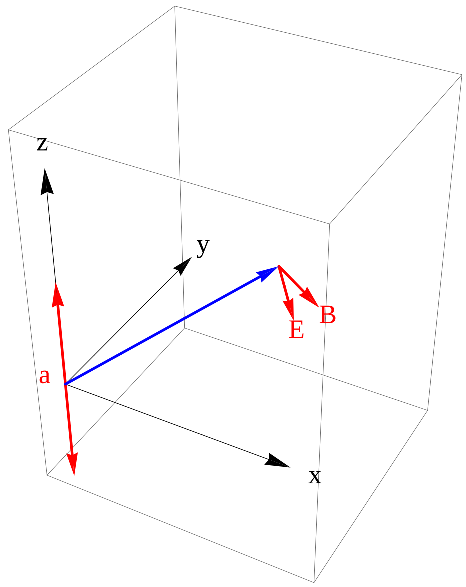



Figure 12.5: Dipole oriented along the z-axis and the associated fields. $\phi$ is the angle between the z-axis and the blue arrrow.

Hence, the total power

$$P = \frac{2}{3}\frac{e^2\mathbf{a}^2}{c^3} = \frac{2}{3}\frac{\ddot{\mathbf{d}}^2}{c^3} \tag{12.54}$$

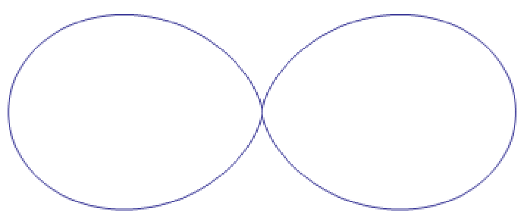


Figure 12.6: Polar plot of the power radiated by a dipole antenna as function of the angle, Eq. 12.51. The maximal power is radiated in the plane perpendicular to the dipole.

The radiation from a dipole antenna is not isotropic: It does not radiate at all in the directions of the dipole. The good news is that you can play with the geometry of the antenna to make it directional so you do not waste power in directions that you do not want.

**Remark 12.4.** *You can not make an isotropic antenna. No matter how complicated an antenna you make the Poynting vector must vanish it at least in two directions. This is a consequence of topology: The vector B is tangent to the* *sphere. It is a fact that every vector field tangent to the sphere must vanish at two points, at least. (Or vanish quadratically at one point.) This is sometimes expressed as* you can not comb a tennis ball*. Hence $P$ must vanish at two points at least.*

## 12.8 Classical instability of atoms

I now want to explain a puzzle in classical electrodynamics that turned out to be a window that opened the way to quantum mechanics. In classical physics atoms are unstable, and should collapse in short time by emitting radiation with diverging frequency.

Consider a charge $e$ in Keplerian orbit around a nucleus of charge $e$ in a circular orbit. The energy (non-relativistic) of the system is

$$E = -\frac{1}{2}\frac{e^2}{|\mathbf{x}|}$$

while the acceleration increases rapidly at the orbit gets smaller:

$$a = \frac{e^2}{m|\mathbf{x}|^2}$$

The rate of loss of energy by radiation is

$$-\dot{E} = \frac{2}{3}\frac{e^2}{c^3}a^2$$

We can now eliminate $a$ and obtain a differential equation for the energy

$$\dot{E} = -kE^4, \quad k = \frac{2^5}{3(me)^2c^3} \tag{12.55}$$

Energy is lost with accelerating rate leading to blow up at finite time. The differential equation is easy to integrate since

$$k = -\frac{\dot{E}}{E^4} = \tfrac{1}{3}\frac{d(E^{-3})}{dt} \tag{12.56}$$

Hence

$$E(t) = E_0\,(1-\gamma t)^{-1/3}, \quad \gamma = -3E_0^3 k > 0$$

In other words, the charge would collapse on the nucleus in finite time $1/\gamma$.

The ground state energy of hydrogen-like atom is,

$$2E_0 = -mc^2\left(\frac{e^2}{\hbar c}\right)^2 = -mc^2\alpha^2$$

and its period is $2\pi\hbar/E_0$. Therefore, the decay time, counted in periods, is

$$\frac{E_0}{2\pi\hbar\gamma} = \frac{1}{128\pi} \times \alpha^{-3} = 6400$$

This means that the electron in hydrogen would fall on the on the proton in $2 \times 10^{-12}$ seconds. The classical world is unprotected against collapse to the nucleus.

The apparent instability of the atoms in classical physics is one of the reasons that lead to quantum mechanics.

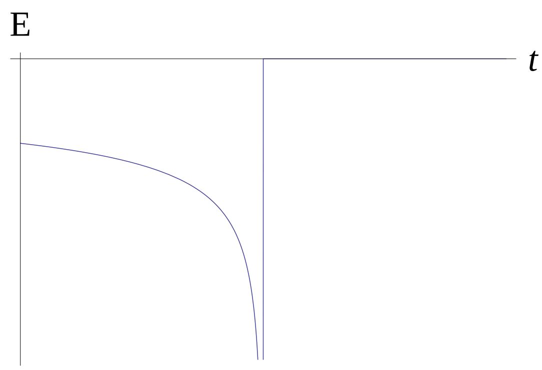



Figure 12.7: Blowup at finite time: The energy of a charged particle encircling the nucleus goes to $-\infty$ in finite time.

# Chapter 13

# Radiation reaction

## 13.1 Is electrodynamics a consistent theory?

### 13.1.1 Physics

Classical electrodynamics is not an accurate description of nature: Nature is quantum mechanical. One can nevertheless ask if classical electrodynamics is a self-consistent theory which is mathematically well defined.

A complete description of a problem in electrodynamics involves Newton equations that describe the motion of charged particles in a given electromagnetic field and Maxwell equations which describe the electromagnetic fields given the 4-currents as sources. As we have seen, both Newton equations for point charges and Maxwell's equations for the fields are consistent with special relativity.

Electrodynamics for point charges has singular source terms $j^\mu$ in Maxwell equations, made from a collection of delta functions. Since the source terms are singular, the fields are also singular. For example, the total field energy of a point charge diverges:

$$\frac{1}{8\pi}\int \frac{e^2}{|\mathbf{x}|^4} d\mathbf{x} = \infty \tag{13.1}$$

The divergence comes from the singularity at $\mathbf{x} = 0$. (The integral at infinity is perfectly convergent in 3 spatial dimensions). As a consequence, when we discussed Maxwell's energy-momentum tensor and the associated conservation of laws, we will encounter diverging quantities.

We can not replace the point charges by rigid small spheres, since rigid spheres are incompatible with the principle of finite propagation speed of special relativity, and involve additional internal degrees of freedom such as rotations, which will affect energy and angular momentum.

If we replace the point charges by other elementary objects such as tiny fluctuating strings, then we cure the disease in total field energy near but we now need to supplement Newton and Maxwell by a theory describing the fluctuations of the strings.

You may wonder if the problem with infinities in not cured by quantum mechanics. After all, Heisenberg uncertainty relation does not allow to fully localize a particle to a point. So, perhaps, a quantum theory of charges coupled with a classical electromagnetic field is a consistent theory? The trouble is that it is not possible to consistently couple a quantum theory for the charges with a classical theory for the electromagnetic fields. This is most easily seen in the Heisenberg picture: In a quantum theory the observables are non-commuting matrices while in a classical theory they are commuting functions. If the two theories are coupled the classical theory will be "infected" by the non-commutativity of the quantum theory.

What about QED: A relativistic quantum theory of both charges and fields? It is now widely believed that this theory although practically useful, is in principle, ill-defined. It is a "phenomenological" mirror of a more involved, hopefully consistent, QCD.

### 13.1.2 Mathematics

Newton equations are non-linear ODE's coupled to Maxwell equations which are linear PDE's. As the the charges are point-like the source terms are singular, and so are the fields. The standard theorems in the theory of differential equations do not cover singular PDE coupled to non-linear ODE. This is worse than the case of Navier-Stokes equations, which is still open.

## 13.2 Non-relativistic interacting particles

There are limiting cases of electrodynamics which are well defined. In particular, this is the case for for non-relativistic point charges. This limit is well defined mathematically[1] and practically useful. The electromagnetic field mediates the interaction between the charges and the limit is encapsulated by the Hamiltonian for mutually interacting particles

$$H = \sum \frac{1}{2m_j}\mathbf{p}_j^2 + \frac{1}{2}\sum_{j\neq k} \frac{e_j e_k}{|\mathbf{x_j} - \mathbf{x_k}|} \tag{13.2}$$

This is the starting point of non-relativistic of atomic physics. To show this we recall first he freedom to impose the Coulomb gauge condition

$$\nabla \cdot \mathbf{A} = 0, \quad \Delta\Phi = -4\pi\rho \tag{13.3}$$

Poisson's equation determines the gauge potential $\Phi$, which responds instantaneously to a change in $\rho$. $\mathbf{A}$ satisfies the wave equation with a source term:

$$-\Delta\mathbf{A} - \ddot{\mathbf{A}} = \frac{4\pi}{c}\mathbf{J} + \nabla\dot{\Phi} \tag{13.4}$$

[1] To be fair, we have thrown the infinite terms associated with self-interaction in Eq. 13.2.

(See section 11.7.)

How much of gauge freedom we still have? We can add

$$\mathbf{A} \mapsto \mathbf{A} + \nabla \times \mathbf{C} \tag{13.5}$$

where $\mathbf{C}$ is a solution of the free wave equation. In particular, we can always impose on $\mathbf{A}$ the condition that it is linear in the source, i.e that it is constructed from the retarded Green function of the wave equation.

When the particles move slowly they are mostly affected by the electric field. This follows from

$$m\dot{u}_\mu + \frac{e}{c} F_{\mu\nu} u^\nu = 0$$

$u^\mu \approx c\delta_0^\mu$, so $F_{\mu\nu}u^\nu \approx F_{\mu 0}c$. By Eq. 13.4 $\mathbf{A}$ is small when when the charges are slow, and the electric field is mostly determined by $\Phi$:

$$\mathbf{E} = -\nabla\Phi - \underbrace{\frac{1}{c}\dot{\mathbf{A}}}_{O(v/c)} \approx -\nabla\Phi \tag{13.6}$$

Maxwell's equations for slow particles, to leading order in $v/c$, reduce to Poisson's equation for $\Phi$. The electromagnetic field is not a dynamical field anymore: It is a slave of the motion of the charges $\rho$. Since we know how Poisson's equation can be explicitly solved given the position of the cahrges, the dynamics is only in the motion of the charges. This gives Eq. 13.2.

## 13.3 Radiation reaction: The Abraham-Lorentz force

Trying to treat Maxwell and Newton self-consistently leads to deep conceptual problems but at the same time, in practice, the effects are normaally rediculously small.

Consider a charged particle moving non-relativistically in a circle due to the action of a central force. We allow for non-electromagnetic forces that make sure that the particle moves in a stationary orbit. The accelerating charge radiates and we want to compute the force that acts back on the charge because of this process.

For circular motion the acceleration is perpendicular to the velocity and

$$\mathbf{a}^2 = (\mathbf{a} \dot{\cdot} \mathbf{v}) - \dot{\mathbf{a}} \cdot \mathbf{v} = -\dot{\mathbf{a}} \cdot \mathbf{v}$$

The radiated power can now be written as

$$P = \frac{2}{3}\frac{e^2}{c^3}\mathbf{a}^2 = -\frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{a}} \cdot \mathbf{v} = -\mathbf{F}_{AL} \cdot \mathbf{v}$$

If the particle is moving at constant speed, an opposite (non-electromagnetic) force to $\mathbf{F_{AL}}$ must be applied to feed back the energy lost to radiation. This can be interpreted as the back reaction of the radiation

$$\mathbf{F}_{AL} = \frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{a}} \tag{13.7}$$

$\dot{\mathbf{a}}$ is known as jerk. It is opposite to the velocity, Fig 13.1 and the force acts abit like friction. This is the Abraham-Lorentz force. It is an unusual force, in two ways. Unlike the usual friction it is not proportional to the velocity but rather to the jerk. Second, it is ultimately a force that a particle applies on itself by shedding radiation.

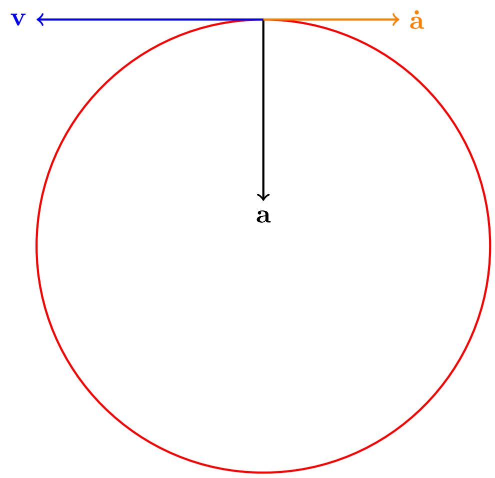



Figure 13.1: In a circular motion every derivative is a rotation by $\pi/2$

**Exercise 13.1** (Covariant form of Abraham Lorentz force)**.** *Using the fact that Newton law*

$$ma_\mu = f_\mu$$

*is consistent with $u \cdot u = -c^2$ provided $f \cdot u = 0$. Using $a \cdot a + u \cdot \dot{a} = 0$, we see that the covariant form of Abaraham-Lorentz force is*

$$ma_\mu = \frac{2e^2}{3c^3}\left(\dot{a}_\mu - \frac{a \cdot a}{c^2}u_\mu\right) \tag{13.8}$$

### 13.3.1 When is radiation reaction important?

To appreciate the significance of the Abraham-Lorentz force consider the ratio

$$\frac{|\mathbf{F}_{AL}|}{m|\mathbf{a}|} = O\left(\frac{e^2}{mc^3}\frac{\dot{a}}{a}\right) = O\left(\tau\omega\right), \quad \tau = \frac{e^2}{mc^3}, \quad \omega = \frac{\dot{a}}{a} \tag{13.9}$$

$\omega$ is the characteristic frequency of the actual motion and $\tau$ is a fundamental time scale associated with radiation reaction. To get some feeling into the meaning of $\tau$ lets express it in terms of the classical radius $r_0$ of a charge particle of mass $m$. This is defined by equating the field energy with the energy in the mass

$$mc^2 = \frac{e^2}{r_0} \tag{13.10}$$

Then $\tau$ is the time it takes light to cross the classical radius:

$$\tau = \frac{r_0}{c} \tag{13.11}$$

For the electron

$$\tau = O(10^{-23})\ [s] \tag{13.12}$$

The ratio in Eq. 13.9 is a ridiculously small number unless the particle has violent jerk, with characteristic frequencies $\omega \ll 10^{23} Hz$.

### 13.3.2 Friction

Small forces can still do something if they act for long time. Consider the equation of motion in a force field $f$ with radiation reaction like friction:

$$\mathbf{a} = \frac{1}{m}\mathbf{f}(\mathbf{x}) + \tau\dot{\mathbf{a}}, \quad \tau = \frac{2}{3}\frac{e^2}{mc^3} \tag{13.13}$$

Viewing $\tau$ as a smallest time scale in the problem we treat it as a perturbation and solve the equation of motion by iteration. To leading order

$$\mathbf{a}_1 = \frac{1}{m}\mathbf{f}(\mathbf{x}_1) \tag{13.14}$$

To second order

$$\begin{aligned}\mathbf{a}_2 &= \frac{1}{m}\mathbf{f}(\mathbf{x}_2) + \tau\dot{\mathbf{a}}_1 \\ &= \frac{1}{m}\left(\mathbf{f}(\mathbf{x}_2) + \tau\dot{\mathbf{f}}(\mathbf{x}_1)\right) \\ &= \frac{1}{m}\left(\mathbf{f}(\mathbf{x}_2) + \tau(\mathbf{v}_1\cdot\nabla)\mathbf{f}(\mathbf{x}_1)\right)\end{aligned} \tag{13.15}$$

which we approximate by an equation with a weak friction term

$$m\mathbf{a} = \mathbf{f}(\mathbf{x}) + \tau(\mathbf{v}\cdot\nabla)\mathbf{f}(\mathbf{x}) \tag{13.16}$$

As an example consider the harmonic oscillator where $f(x) = -m\omega_0^2 x$. The equation of motion is

$$a = -\omega_0^2(x + \tau v) \tag{13.17}$$

which is solved by $x(t) = x(0)e^{i\omega_0 z t}$ where $z$ is a solution of the quadratic equation

$$-z^2 + 1 + iz\tau\omega_0 = 0 \tag{13.18}$$

Since $\omega_0\tau \ll 1$, this is solved by

$$z^2 \approx 1 \pm i\tau\omega \tag{13.19}$$

One solution is decaying in the future and a second solution is exploding in the future. We pick the decaying solution as is self-consistent. The exploding solution is not self-consistent with the assumption that the radiation reaction is a small perturbation.

**Exercise 13.2.** *Write the equations of motion for the Kepler problem with radiation reaction friction term.*

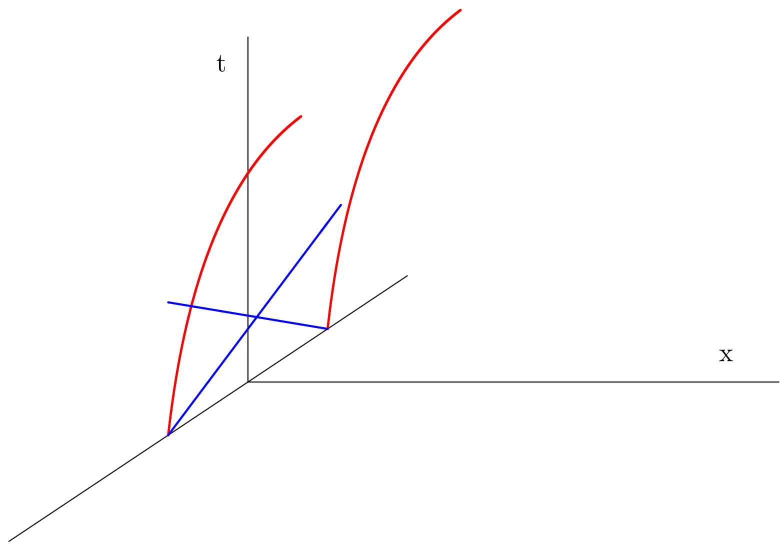
### 13.3.3 The Dumbbell

The derivation of the Abraham-Lorentz force is not completely convincing. We expect a force also when $a$ is constant as the particle still radiates.

To address this consider, following Amos Ori, a the forces that diffeent parts of an extended body apply on each other. In the limit that the size of the object, $\varepsilon \to 0$, we shall recovers Abaraham Lorentz with the extra bonus that we get an interpretation of the mass in terms of the energy of the field.

We shall derive the self-force force on a dumbbell made of two point charges separated by a rod of length $\varepsilon$.

We first compute the forces that the dumbbell applies on itself when it moves in a prescribed way. The world-lines of the dumbbell are shown in the figure (red) and the light-cone is drawn blue. We shall see that as a consequence of retardation, Newton third law is violated, and there is a net force acting on an extended body.




The dumbbell moves along the x-axis and is aligned with the y-axis. So the world line of the two charges is

$$x_{\pm}(t) = (ct, q(t), \pm\varepsilon/2, 0)$$

The two charges communicate when $x_{\pm}(t+\tau) - x_{\mp}(t)$ is light like. The time delay $\tau = O(\varepsilon)$ is small.

$R_{\pm}(\tau) = x_{\pm}(\tau) - x_{\mp}(0)$ is a light like vector. We choose a Lorentz frame so that the dumbbell is at rest at time zero, i.e. $q(0) = \dot{q}(0) = 0$ and so

$$R_{\pm}(\tau) = (c\tau, q(\tau), \pm\varepsilon, 0) \tag{13.20}$$

We want to find the forces at time $\tau$ and express them in terms of the acceleration and jerk *at the same time* $\tau$. For simplicity we take the jerk to be constant. Then

$$a(\tau) = a = a(0) + \dot{a}\tau$$

From this it follows that

$$\dot{q}(\tau) = a(0)\tau + \tfrac{1}{2}\dot{a}\tau^2 = a\tau - \tfrac{1}{2}\dot{a}\tau^2 \tag{13.21}$$

Integrating the velocity gives the position

$$q(\tau) = \tfrac{1}{2}a(0)\tau^2 + \tfrac{1}{6}\dot{a}\tau^3 = \tfrac{1}{2}a\tau^2 + \left(\tfrac{1}{6} - \tfrac{1}{2}\right)\dot{a}\tau^3 = \tfrac{1}{2}a\tau^2 - \tfrac{1}{3}\dot{a}\tau^3 \tag{13.22}$$

This fixes the function $q(\tau)$ in $R_{\pm}$.

The retardation$\tau$ and the dumbbell size $\varepsilon$ are related by the condition that $R$ is light like:

$$(c\tau)^2 = \varepsilon^2 + q^2(\tau)$$

which is a polynomial equation for $\tau$ of order 6. However, as $q(\tau)$ is quadratic in $\tau$. Hence, to leading order

$$c\tau \approx \varepsilon \tag{13.23}$$

The retarded field at time $\tau$ is determined by the velocity and acceleration of the particle at time 0 when the signal was generated:

$$F_{\mu\nu} \quad = \quad -e\left(\frac{R_{[\mu}a_{\nu]}}{(R\cdot u)^2} - \frac{R_{[\mu}u_{\nu]}}{(R\cdot u)^3}\,R\cdot a - c^2\frac{R_{[\mu}u_{\nu]}}{(R\cdot u)^3}\right) \tag{13.24}$$

The four velocity is $u_\mu(0) = (-c, 0, 0, 0)$, and the four acceleration is $a_\mu(0) = \big(0, a(0), 0, 0\big)$. We therefore have from Eq. 13.20

$$R\cdot u = -c^2\tau \approx -c\varepsilon, \quad R\cdot a = -a(0)q(\tau) = O(\varepsilon^2) \tag{13.25}$$

Since this term appears in the denominator of F, the limit $\varepsilon \to 0$ is singular and has to be taken carefully. In particular, we need to keep terms in the numerator to $O(\varepsilon^3)$.

The electric field in the direction of motion on one of the charges due to the other at time $\tau$ is

$$-E_x = F_{01} \quad = \quad -e\left(\underbrace{\frac{R_0 a_1}{(c^2\tau)^2}}_{a_0=0} + \underbrace{\frac{R_1 u_0}{(-c^2\tau)^3}\,R\cdot a}_{u_1=0} + c^2\frac{R_1 u_0}{(-c^2\tau)^3}\right)$$

We are interested in the terms that do not vanish when $c\tau = \varepsilon \to 0$.

By Eqs. 13.20, 13.22, $R_1 = q(\tau) = O(\tau^2)$ the middle terms in the formula above tends to zero with $\tau$ and can be dropped when $\varepsilon \to 0$.

The remaining terms are (to $O(\varepsilon)$)

$$\begin{aligned} E_x = F_{10} \quad &\approx \quad \frac{e}{c^3}\left(-\frac{a(0)}{\tau} + \frac{q(\tau)}{\tau^3}\right) \\ &= \quad \frac{e}{c^3}\left(-\frac{a - \dot{a}\tau}{\tau} + \frac{\frac{1}{2}a - \frac{1}{3}\dot{a}\tau}{\tau}\right) \\ &= \quad -\frac{e}{c^3}\left(\frac{a}{2\tau} - \frac{2}{3}\,\dot{a}\right) \end{aligned} \tag{13.26}$$

The retarded force that one charge applies on the other in the x direction is made of two terms: The leading term diverges as $\varepsilon \to 0$ and is proportional to the acceleration $a$. The subleading term has a finite limit as $\varepsilon \to 0$ and is proportional to the jerk:

$$F = eE = -\underbrace{\frac{e^2}{2\tau c^3}}_{\text{divergent}}\;a + \frac{2e^2}{3c^3}\,\dot{a} \tag{13.27}$$

As expected, in addition to the Abraham-Lorentz force proportional to $\dot{a}$, we find a force proportional to $a$. Newton law for dumbbell with (bare) mass $m_b$, in an external force $F_{ex}$ and radiation self-force is then

$$\left(m_b + \frac{e^2}{2\tau c^3}\right) a = \left(\frac{2e^2}{3c^3}\right) \dot{a} + F_{ex}$$

We interpret the brackets as the effective mass which gets a contribution from the electromagnetic energy in the field.

This is quite tantalizing, for it offers a new interpretation of the mass as being generated by the field. The interpretation is not completely satisfactory for as we send $\tau \to 0$ we need to take the bare mass large and *negative* in order to get a finite effective mass.

**Remark 13.3.** *In this computation we neglected the self-force of the point charge on itself. One can get the self radiation reaction using the following trick Amos Ori taught me. Let $f(e)$ denote this self force. It is a quadratic function of $e$. To find it use the fact the in the limit $\tau \to 0$ the total force on the dumbbell gives an equation for $f(e)$:*

$$F_t = 2f(e) + \frac{4\varepsilon^2}{3c^3}\,\dot{a} = f(2e) = 4f(e)$$

*This says that the radiation reaction force is $f(e) = \frac{2e^2}{3c^3}\,\dot{a}$ as we have seen before.*

## 13.4 Conceptual difficulties

Radiation reaction is conceptually problematic for several reasons. First, it changes the order of Newton equations of motion from second order to third, see Eq. (13.13). This means that fixing the initial position and velocity is not sufficient to determine the orbit. This is in contrast with common experience.

Another problem is that the equations of motion, Eq. (13.13), admit non-physical solutions. For example, with $\mathbf{f} = 0$ the equation is

$$a = \tau \dot{a}\,, \quad \tau = \frac{2e^2}{3mc^3}$$

It admit the solution

$$a(t) = a(0)e^{t/\tau} \Longrightarrow v(t) = v(0) + \frac{a(0)}{\tau}\, e^{t/\tau} \tag{13.28}$$

A particle, initially at rest, self accelerate to large velocities. This can only be avoided if you tune $a(0) = 0$ precisely. In practice, we only tune $x(0)$ and $v(0)$, so how come we do not see these self-accelerations? Classical Electrodynamics for point particles is fundamentally flawed. You could use this to argue for going quantum.